UNIVERSITY OF BATNA 2

FACULTY OF MATHEMATICS AND COMPUTER SCIENCES

DEPARTMENT OF MATHEMATICS

Algebra 2

Lecture notes

by

Zouhir Mokhtari

2021/2022

This is a lecture notes for Linear Algebra 2 written in English and at the end of each chapter equivalent terminologies in French and Arabic are given.

Contents

Chapter 1

Vector spaces

In this chapter, $(\mathbb{K}, +, \times) = \mathbb{R}, \mathbb{C}$ or any commutative field.

1.1 Vector space and vector subspace

Definition 1.1. A vector space over $\mathbb{K}$ is a non-empty set E endowed with two laws:
• an internal composition law called addition and denoted "+", i.e. the application from $E \times E$ to E,
• an external composition law called multiplication by a scalar and denoted by "$\cdot$", i.e. the application from $\mathbb{K} \times E$ to E, such that:

(1) $(E, +)$ is a commutative group, where the neutral element is denoted by 0 or 0_E and the symmetric of an element x of E will be denoted $-x$;

(2) The external law must satisfy for all $x \in E$ and $\alpha, \beta \in \mathbb{K}$: $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;

(3) for all $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$:
$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$;

(4) for all $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$:
$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$.

(5) $1_{\mathbb{K}} \cdot x = x$.

We call the elements of $\mathbb{K}$ "**scalars**" and the elements of E "**vectors**".

Elementary property:
Let E be a $\mathbb{K}$-vector space. Let $x \in E$ and $\alpha \in \mathbb{K}$. So we have:
• $\alpha \cdot x = 0_E$ if and only if $\alpha = 0_{\mathbb{K}}$ where $x = 0_E$;

- $-(\alpha \cdot x) = \alpha \cdot (-x) = (-\alpha) \cdot x$.

Example 1.2. 1. $\mathbb{K}$ is a $\mathbb{K}$-vector space.

2. $\mathbb{R}$ is a $\mathbb{Q}$-vector space.

Example 1.3. We consider $\mathbb{K}^n$ the set of ordered sequences of n elements of $\mathbb{K}$, i.e., $(x_1, x_2, ..., x_n)$ with n being a positive integer. Let $x = (x_1, x_2, ..., x_n)$ and $x' = (x'_1, x'_2, ..., x'_n)$ two elements of $\mathbb{K}^n$ and let $\alpha \in \mathbb{K}$, we set:
$x + x' = (x_1 + x'_1, x_2 + x'_2, ..., x_n + x'_n)$ and $\alpha \cdot x = (\alpha \cdot x_1, \alpha \cdot x_2, ..., \alpha \cdot x_n)$.
Equipped with these two laws, it is easy to verify that $\mathbb{K}^n$ is a $\mathbb{K}$-vector space. In particular, any commutative field $\mathbb{K}$ is a $\mathbb{K}$-vector space as in the previous example. The typical and simplest example in this case, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Example 1.4. The set $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the laws usual ways of adding functions, and multiplying a function by a scalar: $(f + g)(x) = f(x) + g(x)$ and $(\alpha \cdot f)(x) = \alpha \cdot f(x)$, is a $\mathbb{K}$-vector space.

1.1.1 Vector subspace

In this subsection, E will denote a $\mathbb{K}$-vector space.

Definition 1.5. A set F is called a vector subspace of E if

(i) F is a $\mathbb{K}$-vector space.

(ii) $\emptyset \neq F \subset E$.

Proving that a set F is a $\mathbb{K}$-vector subspace from the definition can be quite long. There is another technique to show that a subset F of E is itself a $\mathbb{K}$-vector space.

Theorem 1.6. *A subset F of E is a vector subspace of E if the following conditions hold:*

(i) $0_E \in F$;

(ii) $\forall x, y \in F$, $x + y \in F$;

(iii) $\forall \alpha \in \mathbb{K}$, $\forall x \in F$; $\alpha \cdot x \in F$.

Interpretation:
The conditions of the definition above mean that a non-empty subset F of E is a vector subspace of E if F is stable for addition and for multiplication by a scalar.

Lemma 1.7. *A subset F of E is a vector subspace of E if:*

(i) $(F,+)$ *is a subgroup of* $(E,+)$*;*

(ii) $\forall \alpha \in \mathbb{K}$, $\forall x \in F$; $\alpha \cdot x \in F$.

The following proposition presents a characterization of a vector subspace of E.

Proposition 1.8. *F is a vector subspace of E if and only if F is nonempty and verifies:*

$$\forall x, y \in F, \forall \alpha, \beta \in \mathbb{K}; \alpha \cdot x + \beta \cdot y \in F\,. \tag{1.1.1}$$

Proof. $(\Rightarrow)$ From the definition of vector subspace, the necessary condition is obvious.
$(\Leftarrow)$ Suppose that $F \neq \emptyset$ satisfies the condition (1.1.1) and show that it is a vector subspace of E. For this purpose, we just use lemma 1.7 as follows:
let x and y be two elements of F. For $\alpha = 1$ and $\beta = -1$, we then have $x - y \in F$ which implies $(F,+)$ is a subgroup of $(E,+)$. On the other hand, if we take $y = 0$ in the condition (1.1.1), we obtain (ii) from lemma 1.7. □

Example 1.9. E and $\{0_E\}$ are vector subspaces of E.

Example 1.10. A straight line passing through the origin, a plane passing through the origin are vector subspaces of $E = \mathbb{R}^3$ on $\mathbb{K} = \mathbb{R}$.

Example 1.11. The set $F = \{(x,y) \in \mathbb{R}^2 \mid x - y + 1 = 0\}$ is not a vector subspace of $\mathbb{R}^2$, because the zero vector $0_{\mathbb{R}^2}$ does not belong to F.

Proposition 1.12. *The intersection of two vector subspaces is a vector subspace.*

Proof. Consider F_1 and F_2 two vector subspaces of E. First $0_E \in F_1$, because F_1 is a vector subspace of E. Similarly, $0_E \in F_2$. Thus, $0_E \in F_1 \cap F_2$ and $F_1 \cap F_2$ is therefore not empty. Given $x, y \in F_1 \cap F_2$ and $\alpha, \beta \in \mathbb{K}$, we then have $\alpha \cdot x + \beta \cdot y \in F_1$ since F_1 is a vector subspace of E. Similarly, $\alpha \cdot x + \beta \cdot y \in F_2$. Thus, $\alpha \cdot x + \beta \cdot y \in F_1 \cap F_2$. It follows that $F_1 \cap F_2$ is a vector subspace of E. □

Lemma 1.13. *The intersection of vector subspaces of a vector space E is a vector subspace of E.*

Remark 1.14. In general, the union of two vector subspaces is not a vector subspace (unless one of the two spaces contains the other). Indeed, if we consider $E = \mathbb{R}^2$ and the two vector subspaces $\mathcal{D}_1 = \{(x,y) \in \mathbb{R}^2 \mid y = o\}$ and $\mathcal{D}_2 = \{(x,y) \in \mathbb{R}^2 \mid x = o\}$. Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is not a vector subspace of E. For example, $(\frac{1}{2}, 0) + (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ is the sum of an element of $\mathcal{D}_1$ and an element of $\mathcal{D}_2$, but is not in $\mathcal{D}_1 \cup \mathcal{D}_2$.

1.2 Free families, Generating families, Bases

Notion of linear combination:
A linear combination of vectors $u_1, u_2, ..., u_n$ (with $n \in \mathbb{N}^*$) of a $\mathbb{K}$-vector space E, is a vector which can be written $\sum_{i=1}^{n} \lambda_i \cdot u_i$. The elements $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ are called coefficients of the linear combination.

Example 1.15. Let $u_1, u_2, ..., u_n$; n vectors of a $\mathbb{K}$-vector space E. One can always write 0_E as a linear combination of these vectors, because it suffices to take all zero coefficients.

Remark 1.16. If F is a vector subspace of E, and if $u_1, u_2, ..., u_p \in F$, then any linear combination $\sum_{i=1}^{p} \lambda_i \cdot u_i$ is in F.

Notation 1.17. Given e the vectors $u_1, u_2, ..., u_n$ of a $\mathbb{K}$-vector space E, we denote $Vec(u_1, u_2, ..., u_n)$ the set of linear combinations of $u_1, u_2, ..., u_n$. So we write:

$$Vec(u_1, u_2, ..., u_n) = \left\{ u \in E \mid \exists (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n; u = \sum_{i=1}^{n} \lambda_i \cdot u_i \right\}.$$

Example 1.18. $Vec(0_E) = \{0_E\}$.

Now, we consider a non-empty family $A = (u_1, u_2, ..., u_p)$ of vectors of a $\mathbb{K}$ -vector space E with $p \in \mathbb{N}^*$.

Definition 1.19. We say that A **generates** E, or that it is **generator** of E if and only if $Vec(u_1, u_2, ..., u_p) = E$. In other words, any vector of E is a linear combination of the elements of A.

Definition 1.20. We say that A is **free** if and only if the null vector $\{0_E\}$ is a linear combination of elements of A unique way. In other words:

$$\text{if } \sum_{i=1}^{p} \lambda_i \cdot u_i = 0_E \Rightarrow \forall i \in \{1, \cdots, p\} \, \lambda_i = 0_E \, .$$

Remark 1.21. We can use the following expression:
If A is free then we also say that the vectors $u_1, u_2, ..., u_p$ are linearly independent.

Properties:
1- Any part containing a generating part of E is still a generating part.
2- A family of a single vector is free if and only if this vector is non-zero.
3- Any family that contains the zero vector is not free.
4- Any family contained in a free family is free.
5- We consider a family A of three vectors u_1, u_2, u_3. If two of them are collinear, then the family A is related, but the converse is false.

Definition 1.22. We say that A is a **basis** of a vector subspace F of E if it is **free and generating**. In other words, every vector of F is a linear combination of the elements of A in a unique way. So we have:

$$\forall u \in F, \exists!(\lambda_1, \lambda_2, ..., \lambda_p) \in \mathbb{K}^p; u = \sum_{i=1}^{p} \lambda_i \cdot u_i \, ,$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are called **the coordinates** of the vector u in this basis A, and F is said to be **finite dimension**.

1.3 Vector spaces of finite type

Definition 1.23. A vector space is said to be of finite type if it admits a finite generating family. In other words: if a vector space is generated by a finite family of vectors, it is said to be of finite type.

Theorem 1.24 (Dimension theorem)**.** *In a finite dimensional vector space E, all bases have the same number of elements. This number denoted dim(E) is called the dimension of E.*
Let A be a family of elements of E of finite dimension n. The following properties are equivalent:

(i) A is a basis of E.

(ii) A is free and generates E.

(iii) A is free and of cardinality n.

(v) A is the generator of E and cardinal n.

Remark 1.25. Do not confuse between dimension and cardinal; in a vector space of dimension n, all bases have the same cardinality, but do not speak of the cardinality of a vector space, nor of the dimension of a basis.

Remark 1.26. Practically, we use the above theorem to show that a family A is a basis of E.

Example 1.27. Let $u_1(1, 2), u_2(2, -1)$ be two vectors of the vector space $E = \mathbb{R}^2$ on $\mathbb{K} = \mathbb{R}$. Check that the family $A = (u_1, u_2)$ generates $\mathbb{R}^2$. What can we conclude?
To show that A is a generating family, we look for two real λ_1, λ_2 such that: for all $u(x, y) \in \mathbb{R}^2$, $u = \lambda_1 \cdot u_1 + \lambda_2 \cdot u_2$. After the calculation we will have $\lambda_1 = \frac{1}{5}(x + 2y)$, $\lambda_2 = \frac{1}{5}(2x - y)$. Which means that A generates $\mathbb{R}^2$. On the other hand, it is clear that A is free, of cardinal 2, so A is a basis of $\mathbb{R}^2$.

Corollary 1.28. *Every vector space of finite type admits a finite basis, and all its bases have the same cardinality.*

Corollary 1.29. *In a vector space of dimension n, we have:*
- Any free family has at most n elements.
- Any generating family has at least n elements.

Proposition 1.30 (Characterization of finite-dimensional vector subspaces)**.** *Any vector subspace F of a vector space E of finite type is of finite type, and we have $dim(F) \leq dim(E)$, with equality if and only if $dim(F) = dim(E)$.*

Theorem 1.31 (Incomplete basis theorem)**.** *Let E be a finite dimensional vector space and L a free family of E. Then there exists a basis B of finite cardinality which contains L.*

1.3.1 Rank of a finite family of vectors

Definition 1.32. Let E be a $\mathbb{K}$-vector space and $G = \{v_1, v_2, ..., v_m\}$ a family of m vectors of E. The rank of the family G noted $rank(G)$ is the dimension of the vector subspace $F = Vect(v_1, v_2, ..., v_m)$ generated by the vectors $v_1, v_2, ..., v_m$, ie,

$$rank(G) = dim(F)\,.$$

Properties:
Let E be a $\mathbb{K}$-vector space and $G = \{v_1, v_2, ..., v_m\}$ a family of vectors of E . So we have:
• $0 \leq rank(G) \leq m$.
• If $dim(F) = n$ (finite), then $rank(G) \leq n$.
• $rank(G) = m$ if and only if G is free.
• $rank(G) = 0$ if and only if all vectors of G are zero.

Example 1.33. Let $G = \{v_1 = (2, 3), v_2 = (4, 2), v_3 = (-3, 4)\}$ be a family of the vector space $\mathbb{R}^2$. Determine the rank of G.
It is clear that v_2 and v_3 are linearly independent. On the other hand, by solving the linear system $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0$, we get $2v_1 - v_2 - v_3 = 0$. The family G is therefore dependent. We deduce that $Vect(v_1, v_2, v_3) = Vect(v_2, v_3)$. So $rank(G) = 2$.

1.4 Complementary vector subspace

1.4.1 Sum of two vector subspaces

As we saw earlier, in general the union of two vector subspaces is not a vector subspace. Then, it is useful to know the vector subspaces which contain both these two vector subspaces. Also, especially the smallest of them (of course, in the sense of inclusion).

Definition 1.34. Let F_1 and F_2 be two vector subspaces of a $\mathbb{K}$-vector space E. We call sum of F_1 and F_2 the set noted $F_1 + F_2$, vectors which are the sum of a vector of F_1 and a vector of F_2:

$$F_1 + F_2 = \{u \in E \mid u = u_1 + u_2, u_1 \in F_1, u_2 \in F_2\}.$$

Remark 1.35. We can characterize the vectors u of the sum $F_1 + F_2$, by:

$$u \in F_1 + F_2 \Leftrightarrow \exists (u_1, u_2) \in F_1 \times F_2 \mid u = u_1 + u_2\,.$$

Example 1.36. We consider two intersecting vector lines D_1 and D_2 in the vector space $E = \mathbb{R}^2$. So, it is quite clear that $D_1 + D_2 = \mathbb{R}^2$.

Example 1.37. We have: $F_1 + F_2 = E$ if and only if any vector of E can be written as the sum of a vector of F_1 and a vector of F_2.

Proposition 1.38. *If F_1 and F_2 are two vector subspaces of a $\mathbb{K}$-vector space E, then $F_1 + F_2$ is a vector subspace of E.*

Proof. Consider F_1 and F_2 two vector subspaces of E. First $0_E \in F_1$, because F_1 is a vector subspace of E. Similarly, $0_E \in F_2$. Thus, $0_E = 0_E + 0_E \in F_1 + F_2$ and $F_1 + F_2$ is therefore not empty.
Let $x, y \in F_1 + F_2$ and $\alpha, \beta \in \mathbb{K}$. Since $x \in F_1 + F_2$, there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $x = x_1 + x_2$. So $\alpha \cdot x = \alpha \cdot (x_1 + x_2) = (\alpha \cdot x_1) + (\alpha \cdot x_2) \in F_1 + F_2$, because $\alpha \cdot x_1 \in F_1$ and $\alpha \cdot x_2 \in F_2$. Similarly for $y \in F_1 + F_2$, we get $\beta \cdot y = \beta \cdot (y_1 + y_2) = (\beta \cdot y_1) + (\beta \cdot y_2) \in F_1 + F_2$, because $\beta \cdot y_1 \in F_1$ and $\beta \cdot y_2 \in F_2$ with $y = y_1 + y_2$. It follows that $\alpha \cdot x + \beta \cdot y = (\alpha \cdot x_1 + \beta \cdot y_1) + (\alpha \cdot x_2 + \beta \cdot y_2) \in F_1 + F_2$. □

Proposition 1.39. *If F_1 and F_2 are two vector subspaces of a $\mathbb{K}$-vector space E, then $F_1 + F_2$ is the smallest vector subspace of E containing both F_1 and F_2.*

Proof. We first show that the set $F_1 + F_2$ contains both F_1 and F_2. Indeed, any element $u_1 \in F_1$ is written $u_1 = u_1 + 0_E$ with u_1 belonging to F_1 and 0_E belonging to F_2, because F_2 is a vector subspace of E.
Thereby, u_1 belongs to $F_1 + F_2$. The same for an element of F_2.

Now we show that if H is a vector subspace containing F_1 and F_2, then $F_1 + F_2 \subset H$. As $F_1 \subset H$ we therefore have, if $u_1 \in F_1$ then in particular $u_1 \in H$.

Similarly, if $u_2 \in F_2$ then $u_2 \in H$. Since H is a vector subspace, then $F_1 + F_2 \subset H$. □

1.4.2 Direct sum of two vector subspaces

Definition 1.40. Let F_1 and F_2 be two vector subspaces of a $\mathbb{K}$-vector space E. We say that the sum $F_1 + F_2$ is direct if any vector of $F_1 + F_2$ decomposes uniquely as the sum of an element of F_1 and an element of F_2.

Notation 1.41. When F_1 and F_2 are in direct sum, we write $F_1 + F_2 = F_1 \oplus F_2$.

Remark 1.42. One can characterize the vector subspaces in direct sum, by:

$$F_1 + F_2 \text{ is direct } \Leftrightarrow F_1 \cap F_2 = \{0_E\}\,.$$

Indeed, suppose that the sum $F_1 + F_2$ is direct and $u \in F_1 \cap F_2$. We can then write on the one hand $u = u + 0_E$ with $u \in F_1$ and $0_E \in F_2$, and on the other hand $u = 0_E + u$ with $0_E \in F_1$ and $u \in F_2$. As the sum $F_1 + F_2$ is direct, the decomposition of u following F_1 and F_2 is unique and therefore $u = 0_E$. This proves that $F_1 \cap F_2 \subset \{0_E\}$. Since F_1 and F_2 are two vector subspaces of E, then clearly the inverse inclusion is true.
Conversely, assume that $F_1 \cap F_2 = \{0_E\}$ and show that the sum $F_1 + F_2$ is direct. Suppose we have

$$u = u_1 + u_2 = u_1' + u_2'\,, \tag{1.4.1}$$

with $u_1, u_1' \in F_1$ and $u_2, u_2' \in F_2$. So, $u_1 - u_1' = u_2' - u_2$. Since $u_1 - u_1' \in F_1$ and $u_2' - u_2 \in F_2$, the vector $v = u_1 - u_1' = u_2' - u_2 \in F_1 \cap F_2 = \{0_E\}$. Which implies that $u_1 = u_1'$ and $u_2 = u_2'$. Thus, the writing (1.2) of u is unique, which means that the sum $F + G$ is direct.

Example 1.43. Let H be a subset of a $\mathbb{K}$-vector space E. We can define the vector space generated by H as the sum of the lines generated by the elements of H. For example, if we consider the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $u = (1,1,0)$ of $\mathbb{R}^3$, then it is clear that the vector space generated by $\{e_1, e_2, e_3\}$ is the entire space $\mathbb{R}^3$ while the vector space generated by $\{e_1, e_2, u\}$ is a plane. The subspaces generated by $\{e_1, e_2\}$ and $\{e_3\}$ are in direct sum, but the subspaces generated by $\{e_1, e_2\}$ and by $\{e_2, u\}$ are not in direct sum.

Example 1.44. Two secant lines of the plane $\mathbb{R}^2$ are in direct sum, since their intersection is reduced to the zero vector ($u = 0_{\mathbb{R}^2}$).

Example 1.45. Two secant planes of the space $\mathbb{R}^3$ cannot be in direct sum, since their intersection is a straight line and therefore does not contain only the zero vector ($u = 0_{\mathbb{R}^3}$).

1.4.3 Complementary subspaces

Definition 1.46. Let F_1 and F_2 be two vector subspaces of a $\mathbb{K}$-vector space E. We say that F_1 and F_2 are supplementary in E if the sum $F_1 + F_2$ is direct and if this sum is equal to E.

Remark 1.47. One can characterize the Complementary subspaces, by:

$$F_1 \text{ et } F_2 \text{ are complementary in E } \Leftrightarrow F_1 \cap F_2 = \{0_E\} \text{ et } \ F_1 + F_2 = E\,.$$

In other words: $F_1 \oplus F_2 = E$.

Corollary 1.48. *Let F_1 and F_2 be two vector subspaces of a $\mathbb{K}$-vector space E. So we have:*

$$F_1 \textit{ and } F_2 \textit{ additional in } E \ \Leftrightarrow \forall u \in E, \exists!(u_1, u_2) \in F_1 \times F_2 \mid u = u_1 + u_2\,.$$

Remark 1.49. The notion of spaces in direct sum must not be confused with the notion of additional spaces in another. Indeed, if we consider two intersecting vector lines D_1 and D_2 in the vector space $E = \mathbb{R}^3$ with $D_1 \cap D_2 = \{0_E\}$, and let P be the vector plane that contains them. So, it is clear that $D_1 + D_2 = P$. Which means that their sum is direct and equals exactly the plan $P = D_1 \oplus D_2$. Thus, D_1 and D_2 are supplementary only in P, but not in the whole space.

Remark 1.50. In general, there is no uniqueness of the supplementary. In other words, for a vector subspace F_1 of a $\mathbb{K}$-vector space E, we can find many different supplementary F_2 such as $F_1 \oplus F_2 = E$.

Example 1.51. Let D_1, D_2 and D_3 be three two-by-two secant lines of the vector space $E = \mathbb{R}^2$. So it is easy to see that $D_1 \oplus D_2 = D_1 \oplus D_3 = \mathbb{R}^2$. Which means that D_2 and D_3 are supplements of D_1.

Existence of additional subspaces in finite dimension:

The incomplete basis theorem says that in a finite dimensional vector space, any free family can be completed into a basis of the space. We immediately deduce the existence of supplementary ones.

Proposition 1.52. *Let E be a finite dimensional vector space and F_1 a vector subspace of E. There exists a vector subspace F_2 such that*

$$E = F_1 \oplus F_2 \textit{ et } dim(E) = dim(F_1) + dim(F_2)\,.$$

Theorem 1.53 (Grassmann formula)**.** *If F_1 and F_2 are vector subspaces of E and $F_1 + F_2$ is of finite type, then*

$$dim(F_1 + F_2) = dim(F_1) + dim(F_2) - dim(F_1 \cap F_2)\,.$$

Theorem 1.54 (Characterization of supplementary)**.** *If E is of finite type, then the following conditions are equivalent.*

(i) $E = F_1 \oplus F_2$.

(ii) $F_1 \cap F_2 = \{0_E\}$ *and* $dim(E) = dim(F_1) + dim(F_2)$.

(iii) $E = F_1 + F_2$ *and* $dim(E) = dim(F_1) + dim(F_2)$.

1.5 Terminology translation

English	French	Arabic
Vector	vecteur	
space	éspace	
field	corps	
Identity element	élément neutre	
Integer	entier	
Basis	base	
Scalar	scalair	
Linear combination	combinaison linéaire	
Free	libre	
Rank	rang	
Direct sum	somme direct	

1.6 Exercises

Exercise 1.

For any $x, y \in \mathbb{N}$, define an operation of "addition" by $x+y =$ the maximum of x and y and an operation of "scalar multiplication" by $\alpha \times x = \alpha x$ for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{N}$.

Is $(\mathbb{N}, +, \times)$ a vector space?

Exercise 2.

For any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, define an operation of "addition" by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and an operation of "scalar multiplication" by

$$\alpha \times (x_1, x_2) = (\alpha x_1, 0)$$

for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$.

Is $(\mathbb{R}^2, +, \times)$ a vector space?

Exercise 3.

Let F be the vector space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbb{K} = \mathbb{R}$. Determine which of the following subsets E are subspaces of F. Give reasons for your answers.

1. $E = \{f \in F | f(-x) = f(x), \forall x \in \mathbb{R}\}$, the set of even functions.
2. $E = \{f \in F | f(-x) = -f(x), \forall x \in \mathbb{R}\}$, the set of odd functions.
3. $E = \{f \in F | f(0) = 0\}$.

Exercise 4.

Let E be the set of all functions $f \in F$ that satisfy the differential equation

$$f'' = 0.$$

Show that E is a subspace of F.

Exercise 5.

Let V be a $K-$space with subspaces E_1, E_2. Give an example to show that $E_1 \cup E_2$ may not be a subspace of V.

Exercise 6.

Show that in the space $\mathbb{R}^3$ the vectors $x = (1,1,0), y = (0,1,2)$, and $z = (3,1,-4)$ are linearly dependent.

Exercise 7.
Let $p(x) = x^2 + 2x - 3, q(x) = 2x^2 - 3x + 4$, and $r(x) = ax^2 - 1$. Find the value of a for which the set $\{p, q, r\}$ is linearly independent.

Exercise 8.
Let $u = (1,-1,3), v = (1,0,1)$, and $w = (1,2,c)$ where $c \in \mathbb{R}$. Find the values of c for wich the set $\{u, v, w\}$ is a basis for $\mathbb{R}^3$.

Exercise 9.
Let $\mathbb{P}_2[X]$ be the vector space of a polynomial of degree less than or equal to 2 and $F = \{P_1, P_2, P_3\}$ where:

$$P_1(X) = X^2, P_2(X) = (X-1)^2, P_3(X) = (X+1)^2$$

Show that F is a basis for $\mathbb{P}_2[X]$. Deduce the expression of the polynomial $Q(X) = 12$ in this basis.

Exercise 10.
Let $\mathbb{R}^3$ be the vector space on the field $\mathbb{R}$, $G = [\{(1,1,0), (0,0,1), (1,1,1)\}]$ be a vector subspace of $\mathbb{R}^3$ and let the set F be defined as:

$$F = \left\{(x,y,z) \in \mathbb{R}^3 / 2x + y - z = 0\right\}.$$

1. Show that F is a vector subspace of $\mathbb{R}^3$.
2. Find a basis for each of: $F \cap G, F + G, G, F$ (if any), and give their dimensions.
3. Is $\mathbb{R}^3 = F \bigoplus G$?