Algebra tutorial series 1

Exercise 1 *Which of the following sentences are propositions ? What are the truth values of those that are propositions ?*

1. *Paris is in France or Madrid is in China.*
2. *Open the door.*
3. *The moon is a satelite of the earth.*
4. $x + 5 = 7$.
5. $x + 5 > 9$ *for every real number* x.

Exercise 2 *Determine whether each of the following implications is true or false.*

1. *If* 0.5 *is an integer, then* $1 + 0.5 = 3$.
2. *If* $5 > 2$ *then cats can fly.*
3. *If* $3 \times 5 = 15$ *then* $1 + 2 = 3$.
4. *For any real* $x \in \mathbb{R}$, *if* $x \leq 0$ *then* $(x - 1) < 0$.

Exercise 3 *Are the statements* $P \Rightarrow (Q \vee R)$ *and* $(P \Rightarrow Q) \vee (P \Rightarrow R)$. *logically equivalent ?*
Simplify the following statements.

1. $\overline{(P \Rightarrow \overline{Q})}$.
2. $(\overline{P} \vee \overline{Q}) \Rightarrow \overline{(\overline{Q} \wedge R)}$.
3. $\overline{\left((P \Rightarrow \overline{Q}) \vee \overline{R \wedge \overline{R}}\right)}$

Exercise 4 *Consider the statement "for all integers* a *and* b, *if* $a + b$ *is even, then* a *and* b *are even"*

1. *Write the contrapositive of the statement.*
2. *Write the converse of the statement.*
3. *Write the negation of the statement.*
4. *Is the original statement true or false ? Prove your answer.*
5. *Is the contrapositive of the original statement true or false ? Prove your answer.*
6. *Is the converse of the original statement true or false ? Prove your answer.*
7. *Is the negation of the original statement true or false ? Prove your answer.*

Exercise 5 *Verify whether the following compound propositions are tautologies or contradictions or Contingency*

1. $(P \wedge Q) \wedge \overline{(P \vee Q)}$
2. $\left((P \vee Q) \wedge \overline{P}\right) \Rightarrow Q$
3. $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \Rightarrow Q)$
4. $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$.

Exercise 6 *Complete, when possible, with* $\forall$ *or* $\exists$ *to obtain the strongest true statements.*

1. $\cdots x \in \mathbb{R}, \quad (x + 1)^2 = x^2 + 2x + 1$.
2. $\cdots x \in \mathbb{R}, \quad x^2 + 3x + 2 = 0$.
3. $\cdots x \in \mathbb{R} / \quad 2x + 1 = 0$
4. $\cdots x \in \mathbb{N} / \quad x \leq \pi$

5. $\cdots x \in \mathbb{R}: \quad x^2+2x+3=0$
6. $\cdots x \in \emptyset : 2=3.$

Exercise 7 *Determine among the following propositions which ones are true and write, after justification, the negation of those which are false*

1. $\forall x \in \mathbb{R}^+, x^3 \geq x.$
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x+y>0.$
3. $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x+y>0.$
4. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y>0.$
5. $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, y^2>x.$
6. $\forall x \in \mathbb{R} : \forall y \in \mathbb{R} : x+y>xy.$
7. $\exists (x,y,z) \in \mathbb{R}^3 / |x-y|=3, |y-z|=4$ *and* $|x-z|=8.$
   N.B. *We recall the triangular inequality :* $|a+b| \leq |a|+|b|.$

Exercise 8 *Let* $f:\mathbb{R} \longrightarrow \mathbb{R}$ *a function. Give the negation of :*

1. $\forall M>0, \exists A>0 : \forall x \geq A, f(x)>M.$
2. $\forall x \in \mathbb{R}, f(x)>0 \Rightarrow x \leq 0.$
3. $\forall \epsilon>0, \exists \eta>0 : \forall (x,y) \in I^2, (|x-y| \leq \eta \Rightarrow |f(x)-f(y)| \leq \epsilon).$

Exercise 9 *Soit* $f:\mathbb{R} \longrightarrow \mathbb{R}$ *a function. Express using quantifiers, the following propositions :*

1. *f is constant, f is not constant.*
2. *f is increasing.*
3. *f is bounded.*

Exercise 10
1. *Prove by direct reasoning that if :* $a, b \in \mathbb{Q}$, *then* $a+b \in \mathbb{Q}$.
2. *Prove by contraposition that for all real x and y,* $(x \neq y) \Rightarrow (x+1)(y-1) \neq (x-1)(y+1)$.
3. *Prove by contraposition that, if n^2 is even, then n is even.*
4. *Prove by contradiction that $\sqrt{2}$ is an irrational number.*
5. *Prove by induction that, for all* $n \in \mathbb{N}^*, 2^{n-1} \leq n^! \leq n^n$.

Additional exercises

Exercise 1 ***I.*** *Let P, Q and R three propositions, give the negation of*

1. $P \wedge (\overline{Q} \vee R)$
2. $(P \wedge Q) \Rightarrow R.$

II. *Write the negations of the following propositions*

1. $P: \quad 0<x \leq 1,$
2. $Q: \quad (x^2=1) \Rightarrow (x=1)$

Exercise 2 *Use the truth table method to verify whether the following logical consequences and equivalences are correct :*

1. $\left[(P \Rightarrow (Q \vee R)) \wedge \overline{R}\right] \Vdash P \Rightarrow Q.$
2. $(P \Rightarrow Q) \wedge \overline{Q} \Vdash \overline{P}$
3. $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$
4. $(P \vee Q) \wedge (\overline{P} \Rightarrow \overline{Q}) \Leftrightarrow Q$
5. $(P \wedge Q) \vee R \Leftrightarrow (P \Rightarrow \overline{Q}) \Rightarrow R$
6. $((P \Rightarrow Q) \Rightarrow Q) \Rightarrow Q \Leftrightarrow P \Rightarrow Q$

N.B. F_2 is the logical consequence of F_1, if F_2 is true every time F_1 is true and we write $F_1 \Vdash F_2$

Exercise 3
1. *Let $n>0$. Prove by contradiction that if n is the square of an integer, then $2n$ is not the square of an integer.*
2. *Prove by induction that, for n any positive integer, 6^n-1 is divisible by 5.*
3. *Prove by contradiction that if you arrange $(n+1)$ pairs of socks in n distinct drawers, then there is at least one drawer containing at least 2 pairs of socks.*