Algebra tutorial series 2

Exercise 1 *Let A, B and C be three subsets of a set E. Show that*

1. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
2. $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Where $\overline{A}$ denotes the complement of A in E.
Simplify the following sets :

1. $\overline{A \cup B} \cap \overline{C \cup \overline{A}}$;
2. $\overline{A \cap B} \cup \overline{C \cap \overline{A}}$.

Exercise 2 *Let E be a non-empty set. Show that :*

1. $\forall A, B \in P(E) : A \cup B = A \cap B \Rightarrow A = B$.
2. $\forall A, B, C \in P(E) : (A \cap B = A \cap C \quad \text{and} \quad A \cup B = A \cup C) \Rightarrow B = C$.

Exercise 3 ***Symmetric difference***
Let A and B be two parts of a set E. We call the symmetric difference of A and B, and we denote $A\Delta B$, the set defined by :

$$A\Delta B = (A \cup B) \setminus (A \cap B).$$

1. *Make a drawing, then calculate $A\Delta B$ for $A = \{0, 1, 2, 3\}$ and $\quad B = \{2, 3, 4\}$.*
2. *Show that $A\Delta B = (A \setminus A \cap B) \cup (B \setminus A \cap B)$.*
3. *Determine the sets $A\Delta E, A\Delta A$ and $A\Delta\emptyset$.*
4. *Suppose $A\Delta B = A \cap B$. Prove by contradiction that : $A = \emptyset, (B = \emptyset)$.*
5. *Let $C \in P(E)$. Show that $A\Delta B = A\Delta C$ if and only if $B = C$.*

Exercise 4

1. *What is the image of the sets : $\mathbb{R}, [0, 2\pi], [0, \pi/2]$, and the inverse image of the sets : $[0, 1], [3, 4], [1, 2]$ by the application $x \mapsto \sin x$.*
   Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. Consider the sets $A = [-3, 2], B = [0, 4]$.
2. *Compare the sets $f(A \cap B)$ and $f(A) \cap f(B)$*
3. *What condition must satisfy f so that $f(A \cap B) = f(A) \cap f(B)$.*

Exercise 5 *Let $E = [0, 1], F = [-1, 1], andG = [0, 2]$ be three intervals of $\mathbb{R}$. Consider the map f from E to G defined by : $f(x) = 2 - x$, and the map g from F to G defined by :$g(x) = x^2 + 1$.*

1. *Determine $f(\{1/2\}), f^{-1}(\{0\}), g([-1, 1]), g^{-1}([0, 2])$.*
2. *Is the map f bijective ? justify.*
3. *Is the application g bijective ? justify.*

Exercise 6 *Let $\mathcal{R}$ be the binary relation defined in $\mathbb{R}$ by :*

$$\forall x \in \mathbb{R}, \quad x\mathcal{R}y \Leftrightarrow x^2 - y^2 = x - y.$$

1. *Show that $\mathcal{R}$ is an equivalence relation.*
2. *Calculate the equivalence class of an element x of $\mathbb{R}$.*
3. *Determine the equivalence class of 0, deduce that of 1.*

Exercise 7 *On* $\mathbb{R}^2$, *let* $\prec$ *be the relation given by*

$$\forall (x,y),(x',y') \in \mathbb{R}^2, \quad (x,y) \prec (x',y') \Leftrightarrow x - x' \geq 0 \wedge y = y'.$$

Show that $\prec$ *is an order relation. Is it a total order ?*

Additional exercises

Exercise 1 *Determine the sets* A *and* B *that simultaneously satisfy the following conditions :*

1. $A \cup B = \{1,2,3,4,5\}$,
2. $A \cap B = \{3,4,5\}$,
3. $1 \notin A \setminus B$,
4. $2 \notin B \setminus A$.

Exercise 2 *Let* A *and* B *be two sets. Show that :*

1. $P(A \cap B) = P(A) \cap P(B)$.
2. *Let us show that in general, we do not have* $P(A \cup B) \subseteq P(A) \cup P(B)$.
   (Consider $A = \{0\}, \quad B = \{1\}$*).*

Exercise 3 *Let* f *be a map from* E *to* F, *let* $A \subseteq E$ *and* $B \subseteq F$. *Show that*

1. $A \subseteq f^{-1}(f(A))$, *and* $A = f^{-1}(f(A))$ *for all* $A \subseteq E$ *if and only if* f *is injective.*
2. $f(f^{-1}(B)) \subseteq B$ *and* $B = f(f^{-1}(B))$ *for all* $B \subseteq F$ *if and only if* f *is surjective.*