Course : Algebra 3
Year : 2023/2024
Department of Computer Science

Chapter 1 :

Determinants of matrices

1 Matrices and their properties

Definition 1.1 *A rectangular array that is defined by*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

represents an $m \times n$ matrix where m and n are, respectively, row dimension and column dimension. The elements which are denoted by a_{ij} are real numbers for $1 \preccurlyeq i \preccurlyeq m$ and $1 \preccurlyeq j \preccurlyeq n$.

Example 1.1 *Let A be an $m \times n$ matrix defined by*

$$A = \begin{pmatrix} 4 & 7 \\ 1 & 2 \\ 0 & 8 \end{pmatrix}.$$

Here, we have $m = 3$ and $n = 2$.

Definition 1.2 *, Let $A = [a_{ij}]_{m\times n}$ be an $m \times n$ matrix and Let $B = [b_{ji}]_{n\times m}$ be an $n \times m$ matrix with $b_{ji} = a_{ij}$. Then the matrix B is called the transpose of A and denoted by A^T.*

Example 1.2 *Let A be a 2×3 matrix defined by*

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \end{pmatrix}.$$

The transpose of A is given by

$$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

Remark 1.1 *Let A be an $m \times n$ matrix. A is said to be a square matrix if we have $m = n$*

Definition 1.3 *Assume that A is a square matrix of order n and let a_{ij} be real numbers for all i, j.*

1. *An identity matrix is an $n \times n$ matrix with $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$. This type of matrices is denoted by I_n.*

2. *If the elements of A satisfy $a_{ij} = 0$ for $i \neq j$, then A can be called a diagonal matrix.*

3. *If $a_{ij} = 0$ for all $i \prec j$ (or $i \succ j$), then A represents a so-called lower (or upper) triangular matrix.*

4. *If $a_{ij} = a_{ji}$ for all i, j, then we can say that A is a symmetric matrix.*

Example 1.3 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 5 & 6 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 6 & 0 \\ 1 & 5 \end{pmatrix},$$

and

$$E = \begin{pmatrix} 3 & 8 & 9 & 5 \\ 8 & 1 & 2 & 4 \\ 9 & 2 & 6 & 3 \\ 5 & 4 & 3 & 1 \end{pmatrix}.$$

Thus, we can see that

A is an identity matrix.

B is a diagonal matrix.

C is an upper triangular matrix.

D is a lower triangular matrix.

E is a so-called symmetric matrix.

Theorem 1.1 *Suppose that A, B and C are matrices of order $m \times n$ and that α is a scalar. Then we get that*

a) $A + B = B + A$.

b) $(A + B) + C = A + (B + C)$.

c) $\alpha(A + B) = \alpha A + \alpha B$.

d) $(A + B)^T = A^T + B^T$.

Theorem 1.2 *We assume that A is a matrix of order $m \times n$ and that α is a scalar. Take that B and C are matrices of order $n \times q$, then*

a) $A(B + C) = AB + AC$.

b) $\alpha(AB) = (\alpha A)B$.

c) $(A^T)^T = A$.

d) $(AB)^T = B^T A^T$.

Remark 1.2 *Under the condition that A is a matrix of order $m \times n$, we have*

$$AI_n = I_m A = A$$

2 Calculation of determinants

Definition 2.1 *Assume that A is a square matrix of order n and that the elements of A, denoted by a_{ij}, are real numbers for all i, j. Let $|M_{ij}|$ be the minor of the matrix A corresponding to a_{ij}. Then the determinant of A is given by*

$$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}|, \tag{1}$$

with j is fixed.

Example 2.1 *Consider*

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 7 \\ 3 & 0 & 4 \end{pmatrix}.$$

It is obvious that A and B represent square matrices and their determinants are given by

$$|A| = 4$$

$$\begin{aligned} |B| &= 3\begin{vmatrix} 6 & 7 \\ 0 & 4 \end{vmatrix} - 4\begin{vmatrix} 2 & 7 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix}, \\ &= 106. \end{aligned}$$

Proposition 2.1 *Consider two square matrices of the same order, denoted by A and B, and a scalar α. Assume that a_{ij} are the elements of A. Then the following statements are true.*

a) $|AB| = |A||B|$.

b) $|A| = \prod_{i=1}^{n} a_{ii}$, *in the case where A defines a lower or an upper triangular matrix and in the case where A represents a diagonal matrix of order n.*

c) $|\alpha A| = \alpha^n |A|$.

d) $|A| = |A^T|$.

e) $|I_n + DC| = |I_m + CD|$, *in the case where C is a matrix of order $m \times n$, and where D is a matrix of order $n \times m$.*

f) $|A| = 0$ *in the case where there exists a column or a row of zeros*

3 Invertible matrices

Definition 3.1 *We say that B is the inverse of an $n \times n$ matrix A in the case where B is an $n \times n$ matrix satisfying $BA = I_n$ and $AB = I_n$ such that the inverse of A is represented by A^{-1}.*

Proposition 3.1 *Suppose that A is a square matrix of order n. Then we have*

$$|A|I_n = Aadj(A) = adj(A)A, \tag{2}$$

where

$$adj(A) = C^T, \tag{3}$$

with

$$C_{ij} = (-1)^{i+j}|M_{ij}|. \tag{4}$$

Remark 3.1 *Let A be an $n \times n$ matrix and let $|A| \neq 0$. Thus, we can say that the inverse of A exists and is defined as*

$$A^{-1} = \frac{1}{|A|}adj(A). \tag{5}$$

Example 3.1 *Find the inverse of A in the case where*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and where

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 5 & 6 & 7 \\ 0 & 1 & 2 \end{pmatrix}.$$

1) *Under the condition that $a_{11}a_{22} - a_{21}a_{12} \neq 0$, A^{-1} exists with*

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

2) *We have*

$$\begin{aligned} |A| &= 3\begin{vmatrix} 6 & 7 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 5 & 7 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix}, \\ &= -20. \end{aligned}$$

We note that $|A| \neq 0$, then the inverse of A is

$$A^{-1} = -\frac{1}{20} \begin{pmatrix} 5 & -7 & 22 \\ -10 & 6 & -16 \\ 5 & -3 & -2 \end{pmatrix},$$

such that

$$adj(A) = \begin{pmatrix} 5 & -10 & 5 \\ -7 & 6 & -3 \\ 22 & -16 & -2 \end{pmatrix}.$$

Theorem 3.1 *Assume that A is a matrix and A^{-1} exists. Then we can say that A^{-1} is unique.*

Proof : We suppose that A_1 and A_2 are the inverses of A. Here, we can take

$$A_1 A = I_n, \text{ and } AA_1 = I_n, \tag{6}$$

with

$$A_2 A = I_n, \text{ and } AA_2 = I_n. \tag{7}$$

This leads to

$$\begin{aligned} A_2 &= I_n A_2 \\ &= (A_1 A) A_2 \\ &= A_1 (A A_2) \\ &= A_1 I_n \\ &= A_1 \end{aligned}$$

Then, we deduce that

$$A_2 = A_1, \tag{8}$$

which means that A^{-1} is unique. ∎

Theorem 3.2 *Let A be a square matrix of order n. We can say that the matrix A is invertible if and only if the determinant of A is not equal to zero.*

Proof :

a) Under the assumptions that the inverse of A exists, we get

$$|AA^{-1}| = |A||A^{-1}|. \tag{9}$$

This leads to

$$|I| = |A||A^{-1}|. \tag{10}$$

From the fact that $|I| = 1$ which represents the determinant of a diagonal matrix, we can obtain

$$1 = |A||A^{-1}|. \tag{11}$$

In the sequel, we deduce that

$$|A| \neq 0. \tag{12}$$

b) We let

$$|A| \neq 0, \tag{13}$$

and know

$$|A|I_n = adj(A)A \tag{14}$$

Here, we find

$$A^{-1} = \frac{1}{|A|}adj(A). \tag{15}$$

Then, we can say that the inverse of A exists.

∎

4 Cramer's rule for systems

Consider the system of linear equations

$$AX = B, \tag{16}$$

where A is a square matrix of order n and where B is a column vector. Under the assumption that A is invertible, there exists a unique solution which is denoted by X and defined as $X = A^{-1}B$. Using Cramer's rule, we are able to solve the above system and to compute its solution

$$x_i = \frac{|A_i|}{|A|},\ for\ i = 1, ..., n, \tag{17}$$

such that A_i represents a square matrix, after a change of the $i - th$ column of the matrix A by putting the column vector B.

References

[1] N. Ravishanker and D. K. Dey, A first course in linear model theory, CRC Press, 2001

[2] D. W. Lewis, Matrix theory, World scientific, 1991

[3] V. Sundarapandian, Numerical linear algebra, PHI Learning Private Limited, 2008

[4] Babu Ram, Engineering Mathematics-I (For Wbut), Pearson Education India, 2010

[5] T.K.V. Iyengar and B. Krishna Gandhi and S. Ranganatham and M.V.S.S.N. Prasad , Engineering Mathematics Volume - II (For 2nd Year of JNTU, Anantapur), S. Chand Publishing