Course : Algebra 3
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Department of Computer Science

Chapter 2 :

Linear systems

1 General and basic definitions

Definition 1.1 *Suppose that $v_1, v_2, \cdots, v_n$ is a set of vectors. In order to have that $v_1, v_2, \cdots, v_n$ are linearly dependent, it is necessary that at least one of $\beta_1, \beta_2, \cdots, \beta_n$ is not equal to 0 in the case where*

$$\sum_{i=1}^{n} \beta_i v_i = 0. \tag{1}$$

Example 1.1 *The set of vectors $v_1 = (1, -5)$, $v_2 = (-2, 10)$ is linearly dependent because, for $\beta_1 = 2$ and $\beta_2 = 1$, we get*

$$\beta_1 v_1 + \beta_2 v_2 = (0, 0).$$

Definition 1.2 *Suppose that $v_1, v_2, \cdots, v_n$ is a set of vectors and that*

$$\sum_{i=1}^{n} \beta_i v_i = 0 \Longrightarrow \beta_i = 0, \text{ for all } i = 1, \cdots, n.$$

Then, we can say that the set $v_1, v_2, \cdots, v_n$ is linearly independent.

Example 1.2 *Take that v_1, v_2, v_3, v_4 is the set of vectors where $v_1 = (0, 2, 3, 1)$, $v_2 = (1, 0, 5, 6)$, $v_3 = (3, 4, 0, 5)$ and $v_4 = (1, 2, 3, 0)$. Then, for $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}$, we have*

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4 = (0, 0, 0, 0).$$

This leads to

$$\begin{aligned} \beta_2 + 3\beta_3 + \beta_4 &= 0 \\ 2\beta_1 + 4\beta_3 + 2\beta_4 &= 0 \\ 3\beta_1 + 5\beta_2 + 3\beta_4 &= 0 \\ \beta_1 + 6\beta_2 + 5\beta_3 &= 0 \end{aligned}$$

Here, we get $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$. *Then, we say that* v_1, v_2, v_3, v_4 *are linearly independent.*

Definition 1.3 *Let* A *be an* $m \times n$ *matrix and suppose that*

1. *If there exist zero rows, they must represent the last rows of* A.

2. *In all non-zero rows, we have that the number one is the first non-zero entry. It is also possible to find another non-zero number as the first non-zero entry.*

3. *From top to bottom of* A *, it is necessary to find more of zero numbers which are given before the first non-zero entry.*

Then, A *is called an echelon matrix.*

Definition 1.4 *Suppose that* A *is a square matrix of order* n *and is also nonsingular. Let* r *be the rank of the matrix* A. *Then, we have* $r = n$

Definition 1.5 *Suppose that* A *is an* $m \times n$ *matrix and that* C_j $for\ j = 1, \cdots, n$, *and* R_i $for\ i = 1, \cdots, m$, *are, respectively, columns and rows of the matrix* A. *Let* r *be the maximal number of linearly independent vectors, i.e, columns or rows. Then, we can say that the rank of* A *is equal to* r.

Example 1.3 *Let* A *be a square matrix*

$$A = \begin{pmatrix} 5 & 3 & 0 & 5 \\ 0 & 8 & 5 & 1 \\ 4 & 7 & 3 & 2 \\ 7 & 2 & 1 & 3 \end{pmatrix},$$

Then, the rank of A *is equal to* 4.

Definition 1.6 *Let* A *be an echelon matrix, and let* r *denote the number of non-zero rows of* A. *Then, we can say that the rank of* A *is equal to* r.

2 Non-homogeneous and homogeneous systems

Definition 2.1 *A system of equations that is written in the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

or is given by

$$AX = B, \tag{2}$$

represents a linear system of m equations in n unknowns. There exists only one unknown vector, denoted by X, of the components $x_1, x_2, \cdots, x_n$. To be more precise, the given system (2) is defined by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Definition 2.2 *Let $AX = B$ be a linear system. Then, we say*

1. *$AX = B$ is called a homogeneous system in the case where all the components of the vector B are zeros.*

2. *$AX = B$ is called a non-homogeneous system in the case where B has at least one component that is not equal to zero.*

Definition 2.3 *Let $AX = B$ be a linear system of m equations in n unknowns of the form*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

where A is a so-called coefficient matrix, and where X and B are column vectors. Then, the matrix which is defined by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

is a so-called augmented matrix.

Theorem 2.1 *Let $AX = B$ be a homogeneous system. Then, there exist non-trivial solutions of this system if and only if the rank of the matrix A which is denoted by r satisfies $r \prec n$.*

Theorem 2.2 *Suppose that A is an n-square matrix defined as*

$$A = [v_1, v_2, \cdots, v_n],$$

where $v_1, v_2, \cdots, v_n$ are vectors. Then, we say that $v_1, v_2, \cdots, v_n$ are linearly independent if and only if the matrix A represents a nonsingular matrix.

Theorem 2.3 *Let $AX = B$ be a non-homogeneous system. Then, there exist solutions of this system if and only if the rank of the augmented matrix is equal to the rank of A, denoted by r. To be more precise, we can say that*

1. *there exists a unique solution when $r = n$.*

2. *There exist infinite solutions when $r \prec n$.*

Remark 2.1 *Let $AX = B$ be a linear system, where A is an $m \times n$ matrix, and where X and B are vectors defined, respectively, by*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then, we have the three following cases:

1. *When we take that $m \prec n$, the system $AX = B$ is under-determined.*

2. *When we take that $m = n$, the system $AX = B$ is determined.*

3. *When we take that $m \succ n$, the system $AX = B$ is over-determined.*

Remark 2.2 .

1. $X = (0, 0, \cdots, 0)^T$ *is the trivial solution of a homogeneous system.*

2. *There exist non-trivial solutions of a homogeneous system when $m \prec n$.*

3 Method of Gauss elimination

The linear system of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

can be solved by applying the method of Gauss elimination. For this, we first need to construct the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

Moreover, we check the element a_{11} which must satisfy $a_{11} \neq 0$. We proceed to multiply $1-th$ row by $-a_{i1}/a_{11}$ for $i = 1, \cdots, m$ and to add the results of this operation to the other rows. The application of this procedure $(r-1)$ times leads to get a new matrix.

Remark 3.1 *When we use the method of Gauss elimination, it is necessary to change the position of the equations in the case where $a_{11} = 0$.*

Example 3.1 *Find the solution of this system*

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 1, \\ 2x_1 + 5x_2 + 2x_4 &= 3, \\ 3x_1 + x_3 + 3x_4 &= 1, \\ x_1 + 3x_2 + 4x_3 + 8x_4 &= 2. \end{aligned}$$

The augmented matrix is defined by

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 \\ 2 & 5 & 0 & 2 & 3 \\ 3 & 0 & 1 & 3 & 1 \\ 1 & 3 & 4 & 8 & 2 \end{pmatrix}.$$

The application of the Gauss elimination method leads to get

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -9 & 1 & 0 & -2 \\ 0 & 0 & 4 & 7 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -11 \\ 0 & 0 & 4 & 7 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -11 \\ 0 & 0 & 0 & 7 & 45 \end{pmatrix}.$$

Here, we have

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 1, \\ -x_2 &= 1, \\ x_3 &= -11, \\ 7x_4 &= 45. \end{aligned}$$

Then, the solution of the given system is $x_1 = -17/7, x_2 = -1, x_3 = -11, x_4 = 45/7$. *This means* $X = (-17/7, -1, -11, 45/7)^T$.

4 Solvability of systems by employing matrix inverses and by determinants

Let $AX = B$ be a linear system of m equations in n unknowns. If we suppose that $m = n$. Then, as we show in Chapter 1, it is possible to employ Cramer's rule to solve this system. This method can also be called the method of determinants. On the other hand, it is possible to apply the method of matrix inversion. More precisely, under the condition that A is nonsingular, we can say that there exists a unique solution which is defined by

$$X = A^{-1}B. \tag{3}$$

Theorem 4.1 *Let A denote an n-square matrix and let B be a vector with $B \in \mathbb{R}^n$. Then, we can say that all these statements are equivalent.*

1. *A is a nonsingular matrix.*

2. *When $B = 0$, there exists the trivial solution for the homogeneous system $AX = B$.*

3. *The non-homogeneous system $AX = B$ has solutions.*

4. *There exists a unique solution of the non-homogeneous system $AX = B$.*

5. *The determinant of the matrix A is not equal to zero.*

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