Sets, functions and binary relations

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1 Sets, sets operations

1.0.1 Definitions and notations

Definition 1.0.1 *A set is the mathematical model for a collection of different things (objects) ; a set contains elements or members, which can be mathematical objects of any kind : A class of students, numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets.*

The notation $a \in A$ stands for the statement a belongs to A (a is an element of A). The negation of $a \in A$ is denoted by $a \notin A$.

1. If A is finite, the cardinality of A is the number of its elements denoted by $cardA$.
2. A particular set is the empty set, denoted $\emptyset$ which is the set containing no element.

Here's another way to define sets : a collection of elements that satisfy a property. We then write :

$$E = \{x, P(x)\}.$$

Example 1.0.1

$$\{x \in \mathbb{R} / -1 \leq x \leq 1\} = [-1, 1].$$

Example 1.0.2 *1. The set of (positive, negative and zero) integers by*

$$\mathbb{Z} = \{m - n/m, n \in \mathbb{N}\}. = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

2. The set of rational numbers (ratios of integers) by $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z} \quad and \quad q \neq 0\right\}$.

*3. The set of complex numbers : $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$.
Where we add and multiply complex numbers in the natural way, with the additional identity that $i^2 = -1$, meaning that i is a square root of -1. If $z = x + iy \in \mathbb{C}$, we call x the real part of z and y the imaginary part of z, and we call $|z| = \sqrt{x^2 + y^2}$ the absolute value, or modulus, of z.*

Axiom 1.1 ***(Axiom of extension)*** *Let A and B be sets. Then, $A = B$ if and only if for all x ($x \in A$ if and only if $x \in B$).*

Thus, two sets A and B are equal if they have same members. Two equal sets are treated as same.

1.1 Inclusion

Definition 1.1.1 *Let A and B be sets. We say that A is a* ***subset of*** *B (A is contained in B or B contains A) if every member of A is a member of B. The statement $A$is a subset of B is the same as the statement : For all x(if $x \in A$, then $x \in B$). The notation $A \subseteq B$ stands for the statement A is a subset of B. Thus, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.' The negation of $A \subseteq B$ is denoted by $A \nsubseteq B$ it stands for the statement : There exists $x/ \quad x \in A \quad$ and $\quad x \notin B$).*

1. Every set is a subset of itself, because : For all x(if $x \in A$, then $x \in A$) is a tautology (always a true statement).
2. By convention, for any set E we have $\emptyset \subset E$.

   Cardinality of $\emptyset = |\emptyset| = 0$.
3. If $A \subseteq B$ and $A \neq B$, then we say that A is a proper subset of B. The notation $A \subset B$ stands for the statement A is a proper subset of B.

Proposition 1.1.1 *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

1.2 Power set of E.

Definition 1.2.1 *Let E be a set. We call Power set of E, the set denoted $\mathcal{P}(E)$, defined by :*

$$\mathcal{P}(E) = \{X, X \subseteq E)$$

By definition we have : $\emptyset \in \mathcal{P}(E)$ et $E \in \mathcal{P}(E)$.
For example if $E = \{1, 2, 3\}$: $\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\})$.

Remark 1.2.1
1. *We have $\{a\} \subset E$ and $\{a\} \in \mathcal{P}(E)$.*
2. *If card $E = n$, then card $\mathcal{P}(E) = 2^n$.*
   For $E = \{1, 2, 3\}, \quad \mathcal{P}(E)$ have $2^3 = 8$ parts.
3. *The cardinality of $\mathcal{P}(\emptyset) = 1$.*

Partition
We call partition of a set any family $F \subset E$ such that :

1. The elements of F are disjoint two by two (see example 1.4.1).
2. F is an overlay of E.

Example 1.2.1
1. *$E = \mathbb{N}$, $P = \{2k/ \quad k \in \mathbb{N}\}$, $I = \{2k + 1/ \quad k \in \mathbb{N}\}$.*
   $F = \{P, I\}$ is a partition of E.
2. *$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Then the subsets : $\{0, 1, 2\}$, $\{3, 5, 7\}$, $\{4, 6\}$ et $\{8\}$ constitute a partition of E.*

1.3 Set Complement

Definition 1.3.1 *Let A be a subset of E. We call Set Complement of A in E, and we note $\complement_E^A$, the set of elements of E which do not belong to A.*

$$\complement_E^A = \{x \in E| \quad x \notin A\}.$$

We also note $E \setminus A$ and just $\complement A$ if there is no ambiguity (and sometimes also $A^c or \overline{A}$).

Example 1.3.1 *Let $A = \mathbb{N}$ and $E = \mathbb{Z}$, then $\complement_E^A = \{-x/ \quad x \in \mathbb{N}\}$.*

Let A and B be sets.
there is a unique set defined by $\{x \in B/ \quad x \notin A\}$. This set is denoted by $B - A$, and it B difference A. Clearly, $B - A$ is a subset of A.

1.4 Operations on sets

1.4.1 Intersection

Let A and B be sets. The set $\{x \in A/ \quad x \in B\}$ is denoted by $A \cap B$ and it is called the intersection of A and B. Thus, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Proposition 1.4.1 ***Algebraic properties of the intersection***
Let A, B, C be sets. We havethe relationships :

1. *$A \cap B = B \cap A$ [Commutativity].*
2. *$A \cap B \subseteq A$ and $A \cap B \subseteq B$.*
3. *If $[C \subseteq A \quad and \quad C \subseteq B]$, then $C \subseteq A \cap B$.*
4. *$A \cap A = A$.*
5. *If $A \subset B$, then $A \cap B = A$.*
6. *$A \cap \emptyset = \emptyset$ [absorbent element].*
7. *$A \cap (B \cap C) = (A \cap B) \cap C$. [Associativity] (we can therefore write $A \cap B \cap C$ without ambiguity).*

Example 1.4.1 *If we have $A \cap B = \emptyset$, we say that the sets A and B are **disjoint**.*
We can take as an example :
$]-1, 1] \cap]0, 2] =]0, 1]$, *or also* $\{x \in \mathbb{R}, x^2 \geq 5\} \cap \{x \in \mathbb{R}/ \quad x^2 - 4x + 3 < 0\} = [\sqrt{5}, 3[$.

1.5 Union

For $A, B \subset E$. The set $A \cup B = \{x \in E/ \quad x \in A \quad or \quad x \in B\}$ is called the union of A and B. Thus, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
The "or" is not exclusive : x can belong to A and B at the same time.

Proposition 1.5.1 ***Algebraic properties of the union.***
Let A, B, C be sets. We have the relationships :

1. *$A \cup B = B \cup A$ [Commutativity].*

2. *$A \subseteq A \cup B$ and $B \subseteq A \cap B$.*
3. *If $[A \subseteq C \quad and \quad B \subseteq C]$, then $A \cup B \subseteq C$.*
4. *$A \cup A = A$.*
5. *If $A \subset B$, then $A \cup B = B$.*
6. *$A \cup \emptyset = A$ [Identity].*
7. *$A \cup (B \cup C) = (A \cup B) \cup C$. [Associativity].*

Example 1.5.1 1. *$E = \mathbb{R}, \quad A =]-\infty, 3], \quad B =]-1, 5[$. Then*

$$A \cap B\,]-1, 3], \quad A \cup B =]-\infty, 5[\quad A - B\,]-\infty, -1]$$

2. *$E = \mathbb{R}, \quad A = [-3, 3], \quad B = [0, 1]$. Then*

$$A \setminus B = [-3, 0[\cup]1, 3].$$

$$B \setminus A = \emptyset.$$

3. *We call **symmetric difference** of A and B, and we denote by $A \Delta B$ the set defined by :*

$$A \Delta B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (A \setminus B).$$

We clearly see that the symmetric difference of the sets is not commutative.

1.6 Calculation rules

Let $A, \quad B, \quad C$ parts of a set E. We have :

1. $A \subset B \Leftrightarrow A \cap B = A$.
2. $A \subset B \Leftrightarrow A \cup B = B$.
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $\complement \complement_E^A = A$ and so $A \subset B \Leftrightarrow \complement_E^B \subset \complement_E^A$.
5. $\complement_E^{A \cap B} = \complement_E^A \cup \complement_E^B$.
6. $\complement_E^{A \cup B} = \complement_E^A \cap \complement_E^B$.
   (5 and 6 **Morgan's Laws**).

1.7 Cartesian product of sets

Let E and F be two sets. The Cartesian product, denoted $E \times F$, is the set of all ordered pairs (x, y) where $x \in E$ and $y \in F$. Hence,

$$E \times F = \{(x, y)/ \quad x \in E \wedge y \in F\}$$

Example 1.7.1 $[0, 1] \times \mathbb{R} = \{(x, y)/ \quad 0 \leq x \leq 1, \quad y \in \mathbb{R})$

2 Function

Definition 2.0.1 *A function $f : E \to F$ between sets E, F assigns to each $x \in E$ a unique element $f(x) \in F$. Functions are also called maps, mappings, or transformations.*

A map or function of E in F associates with every element of E a unique element of F denoted $f(x)$.
If f is a map from E to F, and (x, y) an element of $E \times F$ verifying the relation f, we write
$$f: \begin{array}{rl} E \to & F \\ x \mapsto & y \end{array}$$

Example 2.0.1 *The identity function $id_E : E \to E$ on a set E is the function $id_E : x \mapsto x$ that maps every element to itself.*
Let $E = \mathbb{R}^+$ and $F = \mathbb{R}$.
We consider the relation f_1 given by :

$$(x, y) \in E \times F \quad vérifies f_1 \Leftrightarrow y^2 = x$$

For given x there exists $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$, then f is not an application.

1. *The element $f(x) \in F$ is called the image of element $x \in E$ through application f.*
2. *E on which f is defined is called the domain of f and the set F in which it takes its values is called the range. f is an application defined on E with values in F.*
3. ***The graphic** of the application (function) $f : E \to F$ denoted by G_f :*

$$G_f = \{(x, f(x)) \in (E \times F) / x \in E\}.$$

4. ***The equality of applications**. Two applications $f, g : E \to F$ are called equal if and only if they have the same domain, the same codomain, the equality $f = g$ is equivalent to say : for all $x \in E$, $\quad f(x) = g(x)$. We then note $f = g$.*

Definition 2.0.2 ***The range**, or **image**, of a function $f : E \to F$ is the set of values*

$$ranf = \{y \in F : y = f(x) \quad for\ some \quad x \in E\}.$$

*A function is **onto** if its range is all of F ; that is, if for every $y \in F$ there exists $x \in E$ such that $y = f(x)$.*
*A function is **one-to-one** if it maps distinct elements of E to distinct elements of F ; that is, if $x_1, x_2 \in E$ and $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.*
*An onto function is also called a **surjection**, a one-to-one function an **injection**, and a one-to-one, onto function a **bijection**.*

Example 2.0.2 *Consider the maps f, g and h given by :*

$$f: \begin{array}{l} \mathbb{R} \to \mathbb{R}_+ \\ x \mapsto x^2 \end{array}, \qquad g: \begin{array}{l} \mathbb{R}_- \to \mathbb{R} \\ x \mapsto x^2 \end{array}.$$

1. *Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$, then $x_1^2 = x_2^2$ and so $|x_1| = |x_2|$. For $x_1 = -2 \neq x_2 = 2$, we have $f(-2) = f(2) = 4$, then f is an injection.*

2. *For all* $y \in \mathbb{R}_+, \exists x \in \mathbb{R}, x = \sqrt{y}$, *such that* $y = f(x) = x^2$.
   Thus, the map f *is a surjection.*
3. *Let* $x_1, x_2 \in \mathbb{R}_-$ *such that* $g(x_1) = g(x_2)$, *then* $x_1^2 = x_2^2$ *and therefore* $|x_1| = |x_2|$, *i.e.* $-x_1 = -x_2$ *and therefore* $x_1 = x_2$. *Thus* g *is an injection. For* $x_1 = -2 \neq x_2 = 2$,*we have* $f(-2) = f(2) = 4$, *then* f *is not an injection.*
4. *For* $y = -1$, *the equation* $-1 = g(x) = x^2$ *has no solution. Thus, the map* g *is not a surjection.*

Composition and inverses of functions

The successive application of mappings leads to the notion of the composition of functions.

Definition 2.0.3 *The composition of functions* $f : E \to F$ *and* $g : F \to G$, *is the application* $g \circ f : E \to G$ *defined by*

$$g \circ f(x) = g\left(f(x)\right).$$

The order of application of the functions in a composition is crucial and is read from right to left.
The composition $g \circ f$ *can only be defined if the domain of* g *includes the range of* f, *and the existence of* $g \circ f$ *does not imply that* $f \circ g$ *even makes sense.*

Example 2.0.3 *Let* X *be the set of students in a class and* $f : X \to \mathbb{N}$ *the function that maps a student to her age. Let* $g : \mathbb{N} \to \mathbb{N}$ *be the function that adds up the digits in a number e.g.,* $g(1729) = 19$. *If* $x \in X$ *is* 23 *years old, then* $(g \circ f)(x) = 5$, *but* $(f \circ g)(x)$ *makes no sense, since students in the class are not natural numbers. Even if both* $g \circ f$ *and* $f \circ g$ *are defined, they are, in general, different functions.*

Let f be a map from E to F. Then f is bijective if and only if there exists a map g from F to E such that $g \circ f = I_E$ and $f \circ g = I_F$. Further, then $g = f^{-1}$.

Example 2.0.4 1. *Let us define* f, g *thus :*

$$f : \;]0,+\infty[\to \;]0,+\infty[, \quad x \longmapsto \frac{1}{x}.$$

$$g : \;]0,+\infty[\to \mathbb{R}, \quad x \longmapsto \frac{x-1}{x+1}.$$

Then $g \circ f :]0,+\infty[\to \mathbb{R}$ *:*

$$g \circ f\,(x) = g\left(f(x)\right) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = \frac{1-x}{1+x} = -g(x).$$

2.1 Image, Inverse image

Let f be a map from E to F. Let $A \subset E$ and $B \subset F$. The subset

$$f(A) = \{f(x) / \quad x \in A\}.$$

of F is called **the image of** A under the map f.
To say that f is surjective is to say that $f(E) = F$.
The subset

$$f^{-1}(B) = \{x \in E / \quad f(x) \in B\}.$$

of E is called the **inverse image of** B under f.
What are $f^{-1}(F)$ and $f^{-1}(\emptyset)$?

Proposition 2.1.1 *Let f be a map from E to F and $A \subseteq E$. Then $A \subseteq f^{-1}(f(A))$. Also $A = f^{-1}(f(A))$ for all $A \subseteq E$ if and only if f is injective.*

Proof(See tutorial series)

Proposition 2.1.2 *Let f be a map from E to F and $B \subseteq F$. Then $f(f^{-1}(B)) \subseteq B$. Also $B = f(f^{-1}(B))$ for all $B \subseteq F$ if and only if f is surjective.*

Proof(See tutorial series)

Proposition 2.1.3 *Let f be a map from E to F. Let A_1 and A_2 be subsets of E. Then*

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Further, in (ii), equality holds for every pair of subsets A_1 and A_2 of E if and only if f is injective.

Lemme 2.1.1 *Let f be a map from E to F. Let A_1 and A_2 be subsets of E, Let B_1 and B_2 be subsets of F. Then*

$$A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2),$$
$$B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2).$$

Proposition 2.1.4 *Let f be a map from E to F. Let B_1 and B_2 be subsets of F. Then*

1. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
2. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Example 2.1.1 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.*

1. $A =]-2, 2[\qquad f(A) = [0, 4[$
2. $B =]0, 4[\qquad f^{-1}(B) =]-2, 0[\cup]0, 2[$.
3. $C =]-4, 0[\qquad f^{-1}(C) = \emptyset$.

Example 2.1.2 *We consider the map f given by :*

$$\begin{array}{llll} f: & \mathbb{R}^* & \to & \mathbb{R} \\ & x & \longmapsto & 2 + \frac{1}{x^2}. \end{array}$$

1. *Consider $A = [-1, 0[$, then the image of A under the map f is :*

$$f(A) = \left\{ 2 + \frac{1}{x^2} / \quad x \in [-1, 0[\right\} = [3, +\infty[.$$

Indeed, for $-1 \leq x < 0$ we have $2 + \frac{1}{x^2} \geq 3$.

2. *Consider $B = [3, +\infty[$, then the inverse image of B under f is :*

$$f^{-1}(B) = \left\{ x \in \mathbb{R}^* \quad / \quad 2 + \frac{1}{x^2} \in [3, +\infty[\right\} = [-1, 0[\cup]0, 1] .$$

Indeed, for $2 + \frac{1}{x^2} \geq 3$ we have $\dfrac{1}{x^2} \geq 1$ which leads to $x^2 \leq 1$ and so $x \in [-1, 0[\cup]0, 1]$.

Example 2.1.3 *We consider the map* $f : \mathbb{R} \to \mathbb{R}$ *given by :*

$$f(x) = x^2,$$

Let $A = [-1, 4]$.

1. ***The image of*** A ***under*** f ***:***

$$f(A) = f([-1,4]) = \{f(x) \in \mathbb{R} / \quad -1 \leq x \leq 4\}.$$

however $[-1, 4] = [-1, 0] \cup [0, 4]$*, then*
$f([-1,4]) = f([-1,0] \cup [0,4]) = f([-1,0]) \cup f([0,4])$,
It's clear that $-1 \leq x \leq 0 \Rightarrow 0 \leq x^2 \leq 1$ *et* $0 \leq x \leq 4 \Rightarrow 0 \leq x^2 \leq 16.$
So $f([-1,4]) = [0,1] \cup [0,16] = [0,16]$.

2. ***The inverse of*** A ***under*** f***.***

$$f^{-1}([-1,4]) = f^{-1}([-1,0]) \cup f^{-1}([0,4]).$$

however

$$f^{-1}([-1,0]) = \{x \in \mathbb{R} / \quad -1 \leq f(x) \leq 0\} = \{0\},$$

and

$$f^{-1}([0,4]) = \{x \in \mathbb{R} / \quad 0 \leq f(x) \leq 4\}$$

It is clear that $0 \leq f(x) \leq 4 \Leftrightarrow 0 \leq x^2 \leq 4 \Leftrightarrow 0 \leq |x| \leq 2 \Leftrightarrow -2 \leq x \leq 2.$
So

$$f^{-1}([0,4]) = [-2,2]$$

From where

$$f^{-1}([-1,4]) = \{0\} \cup [-2,2] = [-2,2].$$

3 Relations

Definition 3.0.1 *A binary relation* $\mathcal{R}$ *on sets* E *and* F *is a definite relation between elements of* E *and elements of* F*. We write* $x\mathcal{R}y$ *if* $x \in E$ *and* $y \in F$ *are related. If* $E = F$ *, then we call* $\mathcal{R}$ *a relation on* E*.*

Example 3.0.1 *Suppose that* S *is a set of students enrolled in a university and* B *is a set of books in a library. We might define a relation* $\mathcal{R}$ *on* S *and* B *by :*
$s \in S$ *has read* $b \in B$*. In that case,* $s\mathcal{R}b$ *if and only if* s *has read* b*. Another, probably inequivalent relation is :* $s \in S$ *has checked* $b \in B$ *out of the library.*

For sets, it doesn't matter how a relation is defined, only what elements are related.
Let us give some examples to illustrate this definition.

Example 3.0.2 1. *For* $E = \mathbb{R}$*, consider the property* $\mathcal{R}_1$ *defined by :*
(x, y) *check property* $\mathcal{R}_1$ *if* $y = x^2$ *Thus, we do have* $\mathcal{R}_1(2, 4)$ *and* $\mathcal{R}_1(-2, 4)$ *but we do not have* $\mathcal{R}_1(2, 2)$

2. For $E = \mathbb{N}$ consider the property $\mathcal{R}_2(x, y)$ defined by :
(x, y) checks the property $\mathcal{R}_2$ if x divides y, this means that there exists $k \in \mathbb{N}$ such that $y = kx$. Thus, we have $\mathcal{R}_2(0, 0)$ and $\mathcal{R}_2(2, 0)$, but we do not have $\mathcal{R}_2(0, 2)$

In a set E, when a pair (x, y) satisfies a relation $\mathcal{R}$, we write $\mathcal{R}(x, y)$ or $x\mathcal{R}y$.
This last notation is adopted for the following, we then say that : "x is related to y by the relation $\mathcal{R}$".

Example 3.0.3 *Let $P(E)$ be the set of all parts of a set E. We define the relation $\mathcal{R}$ in $P(E)$ by :*

$$\forall A, B \in P(E), \quad A\mathcal{R}B \Leftrightarrow A \subset B,$$

$$\forall A \in P(E), \quad \emptyset \subset A, \quad alors \quad \forall A \in P(E), \quad \emptyset\mathcal{R}A.$$

Definition 3.0.2 *The graph $Gr_{\mathcal{R}}$ of a relation $\mathcal{R}$ on E and F is the subset of $E \times F$ defined by :*

$$Gr_{\mathcal{R}} = \{(x, y) \in E \times F / \quad x\mathcal{R}y\}$$

This graph contains all of the information about which elements are related.

Example 3.0.4 *In $E = \mathbb{R}$, we define the relation $\mathcal{R}$ by :*

$$x\mathcal{R}y \Leftrightarrow x^2 + y^2 < 1$$

Then, $Gr_{\mathcal{R}} = \{(x, y) \in \mathbb{R}^2 / \quad x^2 + y^2 < 1\}$ is the inside of the unit disk.

3.1 Properties of binary relations in a set

Let E a set and let $\mathcal{R}$ a relation defined on E.

3.1.1 The equivalence Relation

The binary relation $\mathcal{R}$ is called

1. **Reflexive**, if $\forall x \in E, \quad x\mathcal{R}x$.
2. **Symmetrical**, if $\forall x, y \in E, \quad x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
3. **transitive**, if $\forall x, y, z \in E, \quad (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.
4. **Anti-symmetrical**, if $\forall x, y \in E, \quad (x\mathcal{R}y) \wedge (y\mathcal{R}x) \Rightarrow x = y$.
5. **Equivalence relation on E**, if it is reflexive, symmetrical and transitive.

Example 3.1.1 *1. Equality in any set is reflexive, symmetrical and transitive.*

2. The inclusion in $P(E)$ is reflexive, non-symmetrical, anti-symmetrical and transitive.

3. In $\mathbb{R}$, the relation "$...\leq...$" is reflexive, non-symmetrical, antisymmetrical and transitive.

3.2 The equivalence relation

Definition 3.2.1 *The binary relation $\mathcal{R}$ on a set E is called equivalence relation if it is réflexive, symmetrical and transitive.*

Example 3.2.1 *In the plane $\mathcal{P}$, the relation "...is parallel..." is an equivalence relation.*

Let $\mathcal{R}$ be an equivalence relation on set E. For each element $x \in E$, the set

$$\overline{x} = \mathcal{R}_x = \{y \in E/ \quad x\mathcal{R}y\}$$

is called the equivalence class of x modulo $\mathcal{R}$ (or in relation to $\mathcal{R}$), and the set $E/\mathcal{R} = \{\overline{x}/ \quad x \in E\}$ is called a factor set (or quotient set) of E through $\mathcal{R}$.

The properties of the equivalence classes. Let $\mathcal{R}$ be an equivalence relation on set E and $x, y \in E$. Then, the following affirmations have effect :

1. $x \in \mathcal{R}_x$,
2. $\mathcal{R}_x = \mathcal{R}_y \Leftrightarrow x\mathcal{R}y \Leftrightarrow y \in \S$
3. $\mathcal{R}_x \neq \mathcal{R}_y \Leftrightarrow \mathcal{R}_x \cap \mathcal{R}_x = \emptyset$,
4. $\sqcup_{x\in E}\mathcal{R}_x = E$.

Partitions on a set. Let E be a non-empty set. A family of subsets $\{E_i/ \quad i \in I\}$ of E is called a partition on E (or of E), if the following conditions are met :

1. $i \in I \Rightarrow E_i \neq \emptyset$,
2. $E_i \neq E_j \Rightarrow E_i \cap E_j = \emptyset$,
3. $\sqcup_{i\in I}E_i = E$.

Théorème 3.2.1 *For any equivalence relation $\mathcal{R}$ on set E, the factor set $E/\mathcal{R} = \{\mathcal{R}_x/ \quad x \in E\}$ is a partition of E.*

Example 3.2.2 *We define on set $E = \mathbb{Z}$ the binary relation $\mathcal{R}$ according to the equivalence*

$$\forall a, b \in E, \quad a\mathcal{R}b \Leftrightarrow \quad \exists k \in \mathbb{Z} : \quad a = b + kn,$$

where $n \in \mathbb{N}^*$, $\quad n$ fixed.

1. *Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.*
2. *Determine the structure of the classes of equivalence.*
3. *Form the factor set $\mathbb{Z}/\mathbb{R}$. Application : $n = 3$.*

We have :

1. *Reflexivity : $\forall a \in \mathbb{Z}, \quad \exists k = 0 \in \mathbb{Z} : \quad a = b + kn = a + 0n$, so $x\mathcal{R}x$.*
2. *Symmetry : $\forall x, y \in \mathbb{Z}, \quad a\mathcal{R}b \Leftrightarrow \exists k \in \mathbb{Z} : \quad a = b + kn$ so $\exists(-k) \in \mathbb{Z}$: $b = a + (-k)n$ and so $b\mathcal{R}a$.*
3. *Transitivity : $\forall a, b, c \in \mathbb{Z}, \quad (a\mathcal{R}b \Leftrightarrow \exists k_1 \in \mathbb{Z} : \quad a = b + k_1n) \wedge (b\mathcal{R}c \Leftrightarrow \exists k_2 \in \mathbb{Z} : \quad b = c + k_2n) \Rightarrow a = b + k_1n = (c + k_2n) + k_1n = c + (k_2 + k_1)n = c + k_3n$ so $a\mathcal{R}c$.*

From 1) - 3) it follows that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.
Let's determine the class of equivalence of an element $x \in E$:
The class of equivalence of $x \in \mathbb{Z}$ will be denoted $\mathcal{R}_x$ or $\overline{x}$ and given by

$$\overline{x} = \{y \in \mathbb{Z} / \quad y\mathcal{R}x\}$$

$$\overline{x} = \{y \in \mathbb{Z} / \quad y = x + kn, \quad k \in \mathbb{Z}\}$$
$$\overline{x} = \{x + kn \in \mathbb{Z} /, \quad k \in \mathbb{Z}\}$$

In the case $n = 3$, let us give the classes of equivalence of $x = 0, x = 1 and \quad x = 2$, their respective classes of equivalence are $\overline{0}, \quad \overline{1} and \quad \overline{2}$ and are given by :
$\overline{0} = \{3k / \quad k \in \mathbb{Z}\}$,
$\overline{1} = \{3k + 1 / \quad k \in \mathbb{Z}\}$,
$\overline{2} = \{3k + 2 / \quad k \in \mathbb{Z}\}$.

Definition 3.2.2 *The relation $\mathcal{R}$ defined above is called a **congruency relation modulo** n **on** $\mathbb{Z}$, and class $\overline{a} = \mathcal{R}_a$ is called a **remainder class modulo** n and its elements are called the representatives of the class.*

The usual notation :

$$a\mathcal{R}b \Leftrightarrow a \equiv b(mode \quad n)$$

(a is congruent with b modulo n),and

$$E/\mathcal{R} = \mathbb{Z}/n\mathbb{Z}.$$

Then

$$E/\mathcal{R} = \mathbb{Z}/n\mathbb{Z} = \left\{\overline{0}, \overline{1}, \cdots, \overline{n-1}\right\}.$$

3.3 Order relations

Definition 3.3.1 *A binary relation $\mathcal{R}$ on the set E is called an order relation on E, if it is reflexive, anti-symmetrical and transitive. Usually, the relation $\mathcal{R}$ is denoted by "$\leq$".*

With this notation, the conditions that "$\leq$" is an order relation on the set E are written :

1. reflexivity $x \in E \Rightarrow x \leq x$;
2. asymmetry $(x \leq y \wedge y \leq x) \Rightarrow x = y$;
3. transitivity $(x \leq y \wedge y \leq z) \Rightarrow x \leq z$.

The pair $(E, \mathcal{R})$, where E is a set and $\mathcal{R}$ an order relation, is called an ordered set.

Definition 3.3.2 *Let $(E, \mathcal{R})$ be an ordered set. The relationship $\mathcal{R}$ is called a **total order** relation if any two elements of E are comparable i.e. For all $x, y \in E$ we have either $x\mathcal{R}y$, or $y\mathcal{R}x$:*

$$\forall x, y \in E, \quad (x\mathcal{R}y \vee y\mathcal{R}x)$$

*We also say that E is totally ordered by the relation $\mathcal{R}$. Otherwise, the order is said to be **partial.***

Example 3.3.1 ***Orders***

1. *A primary example of an order is the standard order* $\cdots \leq \cdots$ *on the natural (or real) numbers. This order is a linear or total order, meaning that two numbers are always comparable.*
2. *Another example of an order is inclusion* $\cdots \subset \cdots$ *on the power set of some set ; one set is " smaller" than another set if it is included in it. This order is a partial order (provided the original set has at least two elements), meaning that two subsets need not be comparable.*
   So, if $E = \{a, b\}$*, the inclusion in* $P(E)$ *is a partial order relation. In fact we have* $\{a\} \nsubseteq \{b\}$ *and* $\{b\} \nsubseteq \{a\}$
3. *On* $\mathbb{R} \times \mathbb{R}$*, we define the relation* $\mathcal{R}$ *by*

$$\forall (x, y), (x', y') \in \mathbb{R} \times \mathbb{R}, (x, y)\mathcal{R}(x', y') \Leftrightarrow ((x \leq x') \wedge (y \leq y'))$$

   It is easy to show that $\mathcal{R}$ *is an order relation. the order is not total order.*
   Indeed, for $(x, y) = (1, 2)$ *and* $(x', y') = (3, 1)$*, we have* $1 \leq 3$ *and* $2 \nleq 1$ *therefore* $(1, 2)$ *is not related to* $(3, 1)$*, similarly we find* $(3, 1)$ *is not related to* $(1, 2)$*.*