Algebra tutorial series 3

Exercise 1 *Let the map* $g : \mathbb{R} - \left\{\frac{1}{2}\right\} \to \mathbb{R}^*$ *be such that :*

$$g(x) = \frac{9}{2x-1}$$

1. *Show that* g *is a bijection. Determine the inverse of* g*.*
2. *Determine* $g^{-1}([-5, 2])$*.*

Exercise 2 ***(Exam January 2023)***
We define on $\mathbb{R}$*, the composition law* $*$ *by :* $\forall x, y \in \mathbb{R}, \quad x * y = x + y - 2.$

1. *Show that* $(\mathbb{R}, *)$ *is an abelian group.*
2. *let* $n \in \mathbb{N}^*$*. We set* $x^{(1)} = x$ *and* $x^{(n+1)} = x^{(n)} * x$

(a) *Calculate* $x^{(2)}, x^{(3)} \quad et \quad x^{(4)}$*.*

(b) *Show that* $\forall n \in \mathbb{N}^* : x^{(n)} = nx - 2(n-1)$*.*

3. *Let* $A = \{x \in \mathbb{R} : x \quad is \quad even\}$*. Show that* $(A, *)$ *is a subgroup of*$(\mathbb{R}, *)$*.*

Exercise 3 *Let* $(G, .)$ *be a group, we denote by* $Z(G) = \{x \in G / \quad \forall y \in G, xy = yx\}$ ***the center*** *of* G*.*

1. *Show that* $Z(G)$ *is a subgroup of* G*.*
2. *Show that* G *is commutative iff* $Z(G) = G$*.*

Note : $xy = x.y$***,*** $(xy)^{-1} = y^{-1}x^{-1}$***,*** e ***the identity element and*** x^{-1} ***the inverse of*** x***.***

Exercise 4 *Let* $*$ *be a binary operation on* $\mathbb{R}^2$ *defined by :*

$$\forall (x, y), (x', y') \in \mathbb{R}^2, \quad (x, y) * (x', y') = (x + x', y + y' + 2xx')$$

1. *Show that* $(\mathbb{R}, *)$ *is an abelian group.*
2. *Show that the curve of equation* $y = x^2$ *is a subgroup of* $(\mathbb{R}^2, *)$ *which we will denote* P*.*
   i.e. $P = \{(x, x^2) / \quad x \in \mathbb{R}\}$ *is a subgroup of* $(\mathbb{R}^2, *)$*.*
3. *Show that the map* $\phi : (\mathbb{R}, +) \to (P, *)$*, defined by* $\phi(x) = (x, x^2)$ *is a group isomorphism*

Exercise 5 *Let* $(A, +, .)$ *be a ring,* 0 *and* 1 *the identities for the laws "* $+$ *" and "."respectively. We define the following operations* $\oplus$ *and* $\otimes$ *on* A *by :*

$$\forall a, b \in A, \quad a \oplus b = a + b + 1 and \quad a \otimes b = a.b + a + b$$

1. *Show that* $(A, \oplus, \otimes)$ *is a ring.*
2. *Show that the map* $f : (A, +, .) \to ((A, \oplus, \otimes)$ *given by* $f(a) = a - 1$ *is an isomorphism of rings.*

Exercise 6 *Let*

$$\mathbb{Z}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} / \quad a, b \in \mathbb{Z}\right\}$$

and

$$\mathbb{Q}\left[\sqrt{2}\right] = \left\{p + q\sqrt{2} / \quad p, q \in \mathbb{Q}\right\}$$

1. *Show that* $\mathbb{Z}\left[\sqrt{2}\right]$ *is a subring of* $\mathbb{R}$*. Is it a field ?*
2. *Show that* $\mathbb{Q}\left[\sqrt{2}\right]$ *is a subfield of* $\mathbb{R}$ *containing* $\mathbb{Q}$ *and that the inclusions* $\mathbb{Q} \subset \mathbb{Q}\left[\sqrt{2}\right] \subset \mathbb{R}$ *are strict.*

3. *Are the following sets subrings of* $\mathbb{Z}\left[\sqrt{2}\right]$ *:*

$$A=\mathbb{Z}, \quad B=\left\{n\sqrt{2}/ \quad n\in\mathbb{Z}\right\}, \quad C=\left\{2m+n\sqrt{2}/ \quad n,m\in\mathbb{Z}\right\}$$

Additional exercises

Exercise 1 *Let $\mathcal{R}$ the binary relation defined on $\mathbb{Z}$ by :*

$$\forall x\in\mathbb{Z}, \quad x\mathcal{R}y \Leftrightarrow x-y \quad \text{is a multiple of 5}$$

1. *Show that $\mathcal{R}$ is an equivalence relation.*
2. *Calculate the equivalence class of an element x of $\mathbb{Z}$.*
3. *Determine the equivalence class of* 0, 1, 2, 3 *and* 4.
4. *Show that* $\mathbb{Z}/5\mathbb{Z}=\left\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4}\right\}$
5. *Show that* $\overline{2023}=\overline{3}$

Exercise 2 *Let $G=\mathbb{R}^*\times\mathbb{R}$, for any elements $(a,b),(c,d)$ of G, we put*

$$(a,b)\otimes(c,d)=(ac,bc+da^2)$$

1. *Calculate* $(-1,1)\otimes(-1,2)$ $(-1,2)\otimes(-1,1)$.
2. *Show that G is a non-abelian group.*

Exercise 3 *Let $(G,*)$ a group, H and K two subgroups of G.*

1. *Show that $H\cap K$ is a subgroup of G.*
2. *Show that $H\cup K$is a subgroup of G if and only if $H\subset K$ or $K\subset H$.*