Algebra tutorial series 3

Exercise 1 *Are the following maps of E into F linear ? If so, determine a basis of the nullspace (kernel) and a basis of the range.*

1. $E = F = \mathbb{R}^2, \forall (x, y) \in \mathbb{R}^2, f(x, y) = (2x + 3y, x)$.
2. $E = F = \mathbb{R}^2, \forall (x, y) \in \mathbb{R}^2, f(x, y) = (y, x + y + 1)$.
3. $E = \mathbb{R}^3, F = \mathbb{R}, \forall (x, y, z) \in \mathbb{R}^3, f(x, y, z) = x + 2y + z$.
4. $E = F = \mathbb{R}^2, \forall (x, y) \in \mathbb{R}^2, f(x, y) = (x + y, xy)$.
5. $E = F = \mathbb{R}, \forall x \in \mathbb{R}, f(x) = x^2$.

Exercise 2 *In each case, give the dimension of the nullspace of f, then the rank of f.*
Is f injective ? surjective ? bijective ?

1. $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (y, z, x)$.
2. $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (x + y, y + z, x - z)$.
3. ***(Left to students)*** $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (2x + my - z, 2x + 2y, x - 2z)$, *according to the value of the parameter m.*

Exercise 3 *Let $\mathbb{R}_4[X]$ be the set of polynomials with real coefficients of degree less than or equal to 4. Show that the map f of $\mathbb{R}_4[X]$ in itself, defined by $f(P) = P - P'$ is linear.*
Is the map f injective ? surjective ?

Exercise 4 *Let*

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. *Compute* $A + I_3, \quad A^2$.
2. *Show that* $A^2 = 2I_3 - A$.
3. *Express* A^{-1} *in terms of* A *and* I_3.
4. *Find* A^{-1} *using the augmented matrix* $[A \mid I]$ *and the reduced echelon form.*

Exercise 5

1. *Find the rank of matrix A by using the row echelon form.*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$$

2. *Represent the following linear system in a matrix equation of the form $Ax = b$.*

$$\begin{cases} -x - 3y + z &= 10 \\ x + y - 3z &= -12 \\ 3x + 4y + 2z &= -5 \end{cases}$$

Solve the above system using the echelon method.