Answer key for tutorial 1. Algebra 2

Exercise 1 ***Question 3. (I.)*** $(E_3, +)$ *a commutative group (to be verified).*

(II.) $\forall u, v \in E_3$, $\forall \alpha, \beta \in \mathbb{R}$,

a./ $\alpha.(u+v) = \alpha.u + \alpha.v$. *Indeed we have :*

$$\begin{aligned}\alpha.(u+v) &= \alpha.((u_n)_n + (v_n)_n) = \alpha.(u_n+v_n)_n = (\alpha \times (u_n+v_n))_n = (\alpha \times u_n + \alpha \times v_n)_n \\ &= (\alpha \times u_n)_n + (\alpha \times v_n)_n = \alpha.(u_n)_n + \alpha.(v_n)_n = \alpha.u + \alpha.v\end{aligned}$$

b./
$$\begin{aligned}(\alpha+\beta).u &= (\alpha+\beta).(u_n)_n = ((\alpha+\beta)\times u_n)_n = (\alpha \times u_n + \beta \times u_n)_n = (\alpha \times u_n)_n + (\beta \times u_n)_n \\ &= \alpha.(u_n)_n + \beta.(u_n)_n = \alpha.u + \beta.u\end{aligned}$$

c./
$$\begin{aligned}(\alpha\times\beta).u &= (\alpha\times\beta).(u_n)_n = ((\alpha\times\beta)\times u_n)_n = (\alpha \times (\beta \times u_n))_n = \alpha.(\beta \times u_n)_n \\ &= \alpha.(\beta.(u_n)_n) = \alpha.(\beta.u)\end{aligned}$$

d./
$$1.u = 1.(u_n)_n = (1 \times u_n)_n = (u_n)_n$$

Exercise 2 ***Question 3.***

1. $\oplus$ ***is commutative.*** *Indeed* *:* $\forall (x,y),(s,t) \in \mathbb{R}^2, (x,y) \oplus (s,t) = (x.s, y.t) = (s.x, t.y) = (s,t) \oplus (x,y)$, *("." is the multiplication in* $\mathbb{R}$*).*
2. $(1,1)$ *is the identity for the operation* $\oplus$. *Indeed :* $\forall (x,y) \in \mathbb{R}^2, (x,y) \oplus (1,1) = (x.1, y.1) = (x,y)$.
3. ***The inverse :*** $\forall (x,y) \in \mathbb{R}^2, \exists^? (x',y') \in \mathbb{R}^2$ *such as :* $(x,y) \oplus (x',y') = (1,1)$, *we have* $(x.x', y.y') = (1,1)$, *then* $x' = \frac{1}{x}$ *and* $y' = \frac{1}{y}$. *The element* $(0,0)$ *has no symmetrical element. Then* $(\mathbb{R}^2, \oplus, \otimes)$ *is not a group, and therefore* $(\mathbb{R}^2, \oplus, \otimes)$ *is not a vector space.*

Exercise 3 $K = \mathbb{R}$.

1. $E = \mathbb{R}^2$

— $(0,0) \notin E_1$, *then* E_1 *is not a subspace of* E.

— E_2 *is a subspace of* E. *Indeed we have :* $2 \times 0 + 3 \times 0 = 0$ *therefore* $(0,0) \in E_2$.
$\forall (x,y),(x',y') \in E_2$ *i.e.* $2x+3y=0$ *and* $2x'+3y'=0$ *then* $\alpha(2x+3y)+\beta(2x'+3y')=0$ *and then* $(\alpha x + \beta x', \alpha y + \beta y') = \alpha(x,y) + \beta(x',y') \in E_2$.

— $(-2,4),(3,-1) \in E_3$ $(xy \leq 0)$ *but* $(-2,4)+(3,-1) = (1,3) \notin E_3$. E_3 *is not a subspace of* E.

— $(1,2) \in E_4$ $(1 \leq 2)$ *but* $(-1)(1,2) = (-1,-2) \notin E_4$. E_4 *is not a subspace of* E.

2. $E = \mathcal{C}(\mathbb{R},\mathbb{R})$ *The set of continuous functions from* $\mathbb{R}$ *into* $\mathbb{R}$.

First, let's note that the null function is an element of E_1.

Let $f, g \in E_1$, $f(0) = f(1)$ *and* $g(0) = g(1)$, *then* $\forall \alpha, \beta \in \mathbb{R}, (\alpha.f + \beta.g)(0) = (\alpha.f)(0) + (\beta.g)(0)$
$= \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha.f)(1) + (\beta.g)(1) = (\alpha.f + \beta.g)(1)$.

Then $\alpha.f + \beta.g \in E_1$.

3. $E = \mathbb{R}_2[X] = \{P = aX^2 + bX + c \ / \ a,b,c \in \mathbb{R}\}$

— $E_1 = \{P \in E \ / \ P'(0) = 2\}$ *is not a subspace of* E *because the null polynomial is not an element of* E_1.

— $E_2 = \{P \in E \ / \ P'(x) \geq 0, \forall x \in \mathbb{R}\}$. *Let* $P \in E_2$ *such that* $P'(x) > 0, \forall x \in \mathbb{R}$, *for* $\alpha < 0$, $\alpha P \notin E_2$ *because* $(\alpha P(x))' = \alpha P'(x) < 0$. *Then* E_2 *is not a subspace of* E.

Vector families

Exercise 4 *Specify whether the following vectors* $\{e_1, \cdots\}$*form a linearly independent or generating family. Express, if possible, the vector "a" as a linear combination of the vectors* e_1, e_2, e_3 *of E, in each of the following cases*

1. $E = \mathbb{R}, e_1 = 3$. $\{e_1\}$ *is linearly independent because* $e_1 \neq 0$.
   $\forall x \in \mathbb{R}, \quad \exists \alpha = \frac{x}{3} \in \mathbb{R}/, \quad x = \alpha e_1 = \frac{x}{3}3$. *Then* $\{e_1\}$ *is a generating family.*
2. $E = \mathbb{R}^2, e_1 = (1, 1), e_2 = (-1, 2), e_3 = (1, 0), \quad a = (2, 4)$.
   Take note *Any* $\mathbb{K}-$*vector space family containing a generating family is itself generating family.*

   $\{e_1, e_3\}$ *is a generating family :* $\forall (x, y) \in \mathbb{R}^2, \quad \exists \alpha, \beta \in \mathbb{R}/, \quad (x, y) = \alpha e_1 + \beta e_2 \Rightarrow x = \alpha - \beta$ *and* $y = \alpha$, *then* $\alpha = y$ *and* $\beta = \alpha - x = y - x$.
   $\{e_1, e_3\}$ *is a generating family, then* $\{e_1, e_2, e_3\}$*is a generating family.*

   card $\{e_1, e_2, e_3\} = 3 > dim\mathbb{R}^2 = 2$, *then* $\{e_1, e_2, e_3\}$ *is not linearly independent.*
   $a = (2, 4) = 4e_1 + 0e_2 - 2e_3$
3.
4. $E = \mathbb{R}^3, e_1 = (1, 1, 0), e_2 = (1, 0, 1), e_3 = (0, 1, 1), \quad a = (1, 1, 1)$.
   $\{e_1, e_2, e_3\}$ *is a generating family :*
   $\forall (x, y, z) \in \mathbb{R}^3, \quad \exists \alpha, \beta, \gamma \in \mathbb{R}/, \quad (x, y, z) = \alpha e_1 + \beta e_2 + \gamma e_3$ *which gives*
   $x = \alpha + \beta$, $y = \alpha + \gamma$ *and* $z = \beta + \gamma$ *and we find* $\gamma = \frac{y+z-x}{2}, \quad \beta = \frac{x+z-y}{2}$ *and* $\alpha = \frac{x+y-z}{2}$.
   Then $\{e_1, e_2, e_3\}$ *is a generating family.*
   $a = (1, 1, 1)$, *we have* $x = y = z = 1, \quad \alpha = \beta = \gamma = 1/2$. *Then* $a = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$.
   $\{e_1, e_2, e_3\}$ *is linearly independent. To be verified.*
5. $E = \mathbb{R}_2[X], \quad e_1 = 1 + 3X, \quad e_2 = X^2 - X, \quad e_3 = X^2 + 1, \quad a = X^2 + X + 1, \quad a = X^3$.
   $\{e_1, e_2, e_3\}$ *is linearly independent :* $\forall \alpha, \beta, \gamma, \in \mathbb{R}, \alpha e_1 + \beta e_2 + \gamma e_3 = 0_{\mathbb{R}_2[x]} \Rightarrow \alpha = \beta = \gamma = 0??$
   $\alpha(1+3X)+\beta(X^2-X)+\gamma(X^2+1) = 0X^2+0X+0 \Rightarrow (\beta+\gamma)X^2+(3\alpha-\beta)X+(\alpha+\gamma) = 0X^2+0X+0$
   which gives : $\beta + \gamma = 0, \quad 3\alpha - \beta = 0$ *and* $\alpha + \gamma = 0$ *and so* $\alpha = \beta = \gamma = 0$.
   $\{e_1, e_2, e_3\}$ *a generating family :* $\forall P = aX^2+bX+C \in \mathbb{R}_2[X], \exists ?? \alpha, \beta, \gamma, \in \mathbb{R}/ \quad P = \alpha e_1+\beta e_2+\gamma e_3$.
   $aX^2 + bX + C = \alpha(1 + 3X) + \beta(X^2 - X) + \gamma(X^2 + 1) = (\beta + \gamma)X^2 + (3\alpha - \beta)X + (\alpha + \gamma)$ *which gives :*
   $\beta + \gamma = a, \quad 3\alpha - \beta = b$ *and* $\alpha + \gamma = c$ *and so*
   $\alpha = (1/2)a + (1/2)b - (1/2)c, \quad \beta = (3/2)a + (1/2)b - (3/2)c$ *and* $\gamma = (3/2)c - (1/2)b - (1/2)a$.
   For $a = X^2 + X + 1, \quad a = b = c = 1$, *then* $\alpha = 1/2, \quad \beta = 1/2$ *and* $\gamma = 1/2$
   i.e. $X^2 + X + 1 = 1/2(1 + 3X) + 1/2(X^2 - X) + 1/2(X^2 + 1)$.
   $a = X^3 \notin \mathbb{R}_2[X]$, *so we can't write a as a linear combination of the vectors* e_1, e_2, e_3.
6. $E = \mathcal{C}(\mathbb{R}, \mathbb{R}), e_1 : x \mapsto x, \quad e_2 : x \mapsto \cos x, \quad e_3 : x \mapsto \sin x$.

   $\{e_1, e_2, e_3\}$ *is linearly independentis :* $\forall \alpha, \beta, \gamma, \in \mathbb{R}, \alpha e_1 + \beta e_2 + \gamma e_3 = 0_{\mathcal{C}(\mathbb{R},\mathbb{R})} \Rightarrow \alpha = \beta = \gamma = 0$.
   Indeed :
   $\alpha e_1 + \beta e_2 + \gamma e_3 = 0_{\mathcal{C}(\mathbb{R},\mathbb{R})} \Leftrightarrow \forall x \in \mathbb{R}, (\alpha e_1 + \beta e_2 + \gamma e_3)(x) = 0_E(x) = 0$ *i.e.* $\forall x \in \mathbb{R}$,
   $\alpha e_1(x) + \beta e_2(x) + \gamma e_3(x) = 0$ *i.e* $\forall x \in \mathbb{R}, \alpha x + \beta \cos(x) + \gamma \sin(x) = 0$. *The equality holds for all x in* $\mathbb{R}$ *then :*
   For $x = 0$, *we have :* $\beta = 0$,
   for $x = \pi/2$ *: we have,* $\alpha\pi/2 + \gamma = 0$,
   for $x = \pi$ *: we have,*$\alpha\pi - \beta = 0$.
   Which gives $\alpha = \beta = \gamma = 0$.
   We can't write $a : x \mapsto \cos^2 x$ *as a linear combination of the vectors* e_1, e_2, e_3,
   then $\{e_1, e_2, e_3\}$ *is not a generating family . a*

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Exercise 5 .

Reminder

— $\forall\lambda \in \mathbb{K} - \{0\}$, $span\{v_1, v_2, \dots, v_i, \dots, v_n\} = span\{v_1, v_2, \dots, \lambda v_i, \dots, v_n\}$.
— $\forall\lambda \in \Bbbk$, $span\{v_1, v_2, \dots, v_i, \dots, v_j, \dots, v_n\} = span\{v_1, v_2, \dots, v_i + \lambda v_j, \dots, v_j, \dots, v_n\}$

$A = \{v_1(2, 0, -1), v_2(3, 2, -4)\}$ $B = \{w_1(1, 2, -3), w_2(0, 4, -5)\}$

$$\begin{aligned} & span\{v_1(2,0,-1), v_2(3,2,-4)\} \\ & \overset{v_2 - v_1 \to v_2}{=} span\{(2,0,-1),(1,2,-3)\} \\ & \overset{-(v_1-2v_2)\to v_1}{=} span\{(0,4,-5),(1,2,-3)\} \\ = \; & span\{w_1(1,2,-3), w_2(0,4,-5)\} = span\{B\} \end{aligned}$$

$A = \{v_1(1, 1, 1, 1), v_2(0, 1, 2, 1), v_3(1, 0, -2, 3), v_4(1, 1, 2, -2)\}$
$\forall\alpha, \beta, \gamma, \delta \in \mathbb{R}$,
$\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0_{\mathbb{R}^4} \iff \alpha(1, 1, 1, 1) + \beta(0, 1, 2, 1) + \gamma(1, 0, -2, 3) + \delta(1, 1, 2, -2) = (0, 0, 0, 0)$
Gives the following system

$$\begin{cases} \alpha + 0\beta + \gamma + \delta = 0 \\ \alpha + \beta + 0\gamma + \delta = 0 \\ \alpha + 2\beta - 2\gamma + 2\delta = 0 \\ \alpha + \beta + 3\gamma - 2\delta = 0 \end{cases} \quad \textit{The solution is : } [\alpha = -2\delta, \beta = \delta, \gamma = \delta].$$

Note that : $2v_1 = v_2 + v_3 + v_4$.
A is not linearly independent, then $rank(A) < 4$
$A' = \{v_1(1, 1, 1, 1), v_2(0, 1, 2, 1), v_3(1, 0, -2, 3)\}$ *is linearly independent, so* $rank(A) = 3$.
$(1, 1, \alpha, \beta) \in span(A) = span(A') \Leftrightarrow \exists\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \;/\; (1, 1, \alpha, \beta) = \lambda_1 v_2 + \lambda_2 v_2 + \lambda_3 v_3$

$$\begin{cases} \lambda_1 + 0\lambda_2 + \lambda_3 = 1 \\ \lambda_1 + \lambda_2 + 0\lambda_3 = 1 \\ \lambda_1 + 2\lambda_2 - 2\lambda_3 = \alpha \\ \lambda_1 + \lambda_2 + 3\lambda_3 = \beta \end{cases}, \; [\alpha = \lambda_1, \;\; \beta = 4 - 3\lambda_1] \; \textit{with } (\lambda_2 = \lambda_3 = 1 - \lambda_1)$$