Answer key for tutorial 1 (continued). Algebra 2

Exercise 1

1. $E = \{(x, y, z) \in \mathbb{R}^3 / \ \ x + z = 0, \quad x + y - 2z = 0\} = \{(x, y, z) \in \mathbb{R}^3 / \ z = -x, \ y = -3x\} = \{(x, -3x, -x) \in \mathbb{R}^3 / \ \ x \in \mathbb{R}\} = \{x\,(1, -3, -1) / \ \ x \in \mathbb{R}\}$
   $E = Vect\,\{w = (1, -3, -1)\}$.
   $\{w = (1, -3, -1)\}$ *is linearly independent because* $(1, -3, 1) \neq (0, 0, 0)$. *Then* $dimE = 1$.
2. $F = span\,\{u = (1, -2, 1), v = (2, 0, 1)\}$ *is linearly independent :*
   $\forall \alpha, \beta \in \mathbb{R} \ / \ \alpha u + \beta v = 0_{\mathbb{R}^3} \Longrightarrow \alpha\,(1, -2, 1) + \beta\,(2, 0, 1) = (0, 0, 0)$
   then $(\alpha + 2\beta, -2\alpha, \alpha + \beta) = (0, 0, 0) \Longrightarrow \alpha = \beta = 0$. *Then* $\{u, v\}$ *is a basis of* F *and so* $\dim F = 2$.
3. $E \cap F = \{(x, y, z) \in \mathbb{R}^3 / \ \ (x, y, z) \in E \wedge (x, y, z) \in F\}$.
   Let $a = (\ x, y, z) \in E \cap F$, *then* $a \in E$ *and* $a \in F$, *we have* $\begin{cases} a \in E \Longleftrightarrow \exists x \in \mathbb{R}/a = (x, -3x, -x) \\ a \in F \Longleftrightarrow \exists \alpha, \beta \in \mathbb{R}/a = \alpha u + \beta v \end{cases}$
   $a = \alpha\,(1, -2, 1) + \beta\,(2, 0, 1)$ *then* $(x, -3x, -x) = (\alpha + 2\beta, -2\alpha, \alpha + \beta)$

   $x = \alpha + 2\beta, \ \ -3x = -2\alpha$ *and* $\ -x = \alpha + \beta$ *i.e.* $\alpha + 2\beta = -\alpha - \beta$ *and* $\ -2\alpha = 3(\alpha + \beta)$ *then* $\alpha = \beta = 0$.
   $E \cap F = \{(0, 0, 0)\}$ *and* $dim(E \cap F) = 0$.

4. $E + F = \{X + X' / \ X \in E$ *and* $X' \in F\} = span\,\{w, u, v\}$.
   We can verify that $\{w, u, v\}$ *is linearly independent then* $\{w, u, v\}$ *is a basis of* $E + F$.
   $dimE + F = 3$.
   We have also $dim(E + F) = dim(E) + dim(F) - dim(E \cap F) = 1 + 2 - 0 = 3$

5. $\mathbb{R}^3 = E + F$ *Indeed :*
   $E + F \subset \mathbb{R}^3$ *(because* $E \subset \mathbb{R}^3$ *and* $F \subset \mathbb{R}^3$, *) and* $dim\mathbb{R}^3 = dim(E + F)$ *then* $\mathbb{R}^3 = E + F$.
   We have $\mathbb{R}^3 = E + F$ *and* $E \cap F = \{(0, 0, 0)\}$ *then* $\mathbb{R}^3 = E \oplus F$.

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$u = 2(1, -3, -1) = 2w \in E$, *then the coordinate of the vector* u *in the basis* $\{w\}$ *is* 2.

Exercise 2

$E = \mathbb{R}_2[X], P_1 = X + 1, P_2 = X^2 - 1, P_3 = X^2 - 2X + 1$.
$\{P_1, P_2, P_3\}$ *is linearly independent :* $\forall \alpha, \beta, \gamma, \in \mathbb{R}, \alpha P_1 + \beta P_2 + \gamma P_3 = 0_{\mathbb{R}_2[x]} \Rightarrow \alpha = \beta = \gamma = 0$
We have : $\alpha(X+1)+\beta(X^2-1)+\gamma(X^2-2X+1) = 0X^2+0X+0 \Rightarrow (\beta+\gamma)X^2+(\alpha-2\gamma)X+(\alpha-\beta+\gamma) = 0X^2 + 0X + 0$, *which gives :*
$\beta + \gamma = 0, \quad \alpha - 2\gamma = 0$ *and* $\alpha - \beta + \gamma = 0$ *and so* $\alpha = \beta = \gamma = 0$.
$\{P_1, P_2, P_3\}$ *a generating family :* $\forall P = aX^2 + bX + C \in \mathbb{R}_2[X], \exists \alpha, \beta, \gamma, \in \mathbb{R} / \quad P = \alpha P_1 + \beta P_2 + \gamma P_3$
Indeed :
$aX^2+bX+C = \alpha(X+1)+\beta(X^2-1)+\gamma(X^2-2X+1) = (\beta+\gamma)X^2+(\alpha-2\gamma)X+(\alpha-\beta+\gamma) = aX^2+bX+c$, *which gives :*
$\beta + \gamma = a, \quad \alpha - 2\gamma = b$ *and* $\alpha - \beta + \gamma = c$ *and so*
$\alpha = (1/2)a + (1/2)b + (1/2)c, \quad \beta = (3/4)a + (1/4)b - (1/4)c$ *and* $\gamma = (1/4)a - (1/4)b + (1/4)c$.

For $P_4 = -X^2 + X + 6, \quad a = -1, \quad b = 1, \quad c = 6$, *then* $\alpha = 3, \quad \beta = -2$ *and* $\gamma = 1$
i.e. $-X^2 + X + 6 = 3(X + 1) - 2(X^2 - 1) + (X^2 - 2X + 1)$.

Exercise 3 $F = \{(x, y, z) \in \mathbb{R}^3 / \quad x - y + z = 0\}$ *and* $G = \{(x, x, x) / \quad x \in \mathbb{R}\}$.
$\mathbb{R}^3 = F \oplus G$. *Indeed :*

1. $F=\{(y+z,y,z)/\quad y,z\in\mathbb{R}\}=\{y(1,1,0)+z(1,0,1)/\quad y,z\in\mathbb{R}\}=span\,\{u=(1,1,0),v=(1,0,1)\}$. *We check that* $\{u,v\}$ *is linearly indepndent, then we conclude that* $\{u,v\}$ *is a basis of* F *and so* $dimF=2$.
2. $G=span\,\{w=(1,1,1)\}$. *and* $\{w=(1,1,1)\}$ *is linearly independent because* $(1,1,1)\neq(0,0,0)$. *Then* $dimG=1$
3. *we show that* $F\cap G=\{(0,0,0)\}$.

   Let $a=(\ x,y,z)\in F\cap G$, *then* $a\in F$ *and* $a\in G$, *we have* $\begin{cases} a\in F\Longleftrightarrow \exists y,z\in\mathbb{R}/a=(y+z,y,z)\\ a\in F\Longleftrightarrow \exists x\in\mathbb{R}/a=(x,x,x)\end{cases}$

   then $(x,x,x)=(y+z,y,z)$

   we find $x=y=z=0$, *so* $F\cap G=\{(0,0,0)\}$ *and* $dimF\cap G=0$.
4. $\mathbb{R}^3=F+G$ *Indeed :*

   $F+G\subset\mathbb{R}^3$ *because* $F\subset\mathbb{R}^3$ *and* $G\subset\mathbb{R}^3$, *and*

   $dim(F+G)=dimF+dimG-dimF\cap G=2+1-0=3=dim\mathbb{R}^3$, *then* $\mathbb{R}^3=F+G$.

   We have $\mathbb{R}^3=F+G$. *and* $F\cap G=\{(0,0,0)\}$ *then* $\mathbb{R}^3=F\oplus G$.

Exercise 4

1. $span(u,v)=\{\alpha(1,-2,4,1)+\beta(1,0,0,2)/\quad \alpha,\beta\in\mathbb{R}\}$
   $=\{(\alpha+\beta,-2\alpha,4\alpha,\alpha+2\beta)/\quad \alpha,\beta\in\mathbb{R}\}$.
2. $B=\{u=(1,-2,4,1),v=(1,0,0,2),e_1=(1,0,0,0),e_3=(0,0,1,0)\}$ *is linearly independent (to be verified) then* B *is a basis of* $\mathbb{R}^4$. *(* $card(B)=4=dim\mathbb{R}^4$ *).*
3. $\{u=(2,3,5),v=(4,6,10),w=(-2,-3,-5)\}$ *is not linearly independent, then* $rank(\{u,v,w\}<4$.

   $\{u=(2,3,5),v=(4,6,10)\}$*is not linearly independent,* $(v=2u)$,

   $\{u=(2,3,5),w=(-2,-3,-5)\}$ *is not linearly independent,* $(w=-u)$ *and*

   $\{v=(4,6,10),w=(-2,-3,-5)\}$ *is not linearly independent,* $(v=-2w)$.

   Then $rank(\{u,v,w\}<2$.

   $\{v=(4,6,10)\}$ *(for example) is linearly independent,* $(v\neq 0)$, *then* $rank(\{u,v,w\}=1$.