Chapter 3

Introduction to the Calculation of Probabilities

0.1 Introduction

Experiments play a critical role in science and engineering, offering valuable insights because they can be replicated under nearly identical conditions to produce consistent results. This control over variables influencing the experiment's outcome allows for predictable findings.

However, there are experiments where it is not possible to determine or manage all variables, leading to variations in results even under similar conditions. Such experiments are termed random. Here are a few examples.

Example 1. 1. *If we toss a coin, the result of the experiment is that it will either come up "tails," symbolized by* T *(or 0), or "heads," symbolized by* H *(or 1), i.e., one of the elements of the set* $\{H, T\}$ *(or* $\{0, 1\}$*). If we toss a coin twice, there are four results possible, as indicated by* $\{HH, HT, TH, TT\}$*, i.e., both heads, heads on the first and tails on the second, etc.*

2. *If we are testing light bulbs for functionality, the outcome of each test can either be that the bulb is functional or non-functional. Thus, when a light bulb is tested, it will be a member of the set {functional, non-functional}.*

0.2 Definitions and Examples

0.2.1 Sample Spaces

Definition 1. *A set S, consisting of all possible outcomes of a random experiment, is referred to as a sample space.*

Example 2. *If we toss a die, one sample space, or set of all possible outcomes, is given by $S = \{1, 2, 3, 4, 5, 6\}$*

0.2.2 Events

Definition 2. *An event is a subset A of the sample space S, meaning it comprises a set of possible outcomes. If the outcome of an experiment is an element of A, we say that the event A has occurred. An event that consists of a single point in S is frequently referred to as a simple or elementary event.*

Example 3.

1. *Consider the experiment of rolling a six-sided die. The sample space S for this experiment is $\{1, 2, 3, 4, 5, 6\}$, representing all possible outcomes. Let us define an event A as rolling an even number. Therefore, A is the subset of S and can be expressed as $\{2, 4, 6\}$. If the outcome of rolling the die is an element of A (i.e., the die shows a 2, 4, or 6), then we say that the event A has occurred. This event, consisting of multiple possible outcomes, exemplifies a non-elementary event.*

2. *Consider the experiment of drawing a single card from a standard 52-card deck. The sample space S for this experiment includes all 52 cards. Define an event B as drawing a card from the hearts suit. Thus, B is a subset of S and consists of the set $\{Ace\,of\,Hearts, 2\,of\,Hearts, 3\,of\,Hearts, \ldots, King\,of\,Hearts\}$. If a card drawn is an element of B (i.e., any card from the hearts suit), then we say that the event B has occurred. This event demonstrates a specific, non-elementary event based on the suit of the card.*

Remark 0.2.1. *As particular events, we have*

- *S itself, which is the sure or certain event.*

- *ϕ the empty set , which is called the impossible event.*

0.2.3 Operations on Events in Probability Theory

By applying set operations on events in a sample space S, we can derive new events within S. For instance, if A and B are events, then:

1. $A \cup B$ is the event "either A or B, or both." $A \cup B$ is called the union of A and B.

2. $A \cap B$ is the event "both A and B." $A \cap B$ is called the intersection of A and B.

3. A^c or $\bar{A}$ is the event "not A." A^c is called the complement of A.

4. $A \setminus B$ is the event "A but not B." Specifically, $A \setminus S = A^c$.

Remark 0.2.2.

- *If the sets corresponding to events A and B are disjoint, denoted as $A \cap B = \emptyset$, the events are often said to be mutually exclusive. This implies that they cannot both occur.*

- *A collection of events $A_1, A_2, \ldots, A_n$ is mutually exclusive if every pair in the collection is mutually exclusive.*

Example 4. *Referring to the experiment of tossing a coin twice, let A be the event "at least one head occurs" and B the event "the second toss results in a tail." Then A consists of the outcomes $\{HT, TH, HH\}$, and B consists of the outcomes $\{HT, TT\}$. Therefore, we have:*

- $A \cap B = \{HT\}$

- $A^c = \{TT\}$

- $A \cup B = \{HT, TH, HH, TT\} = S$

0.2.4 Basic Laws of Probability

Suppose we have a sample space S and $P(A)$ is the probability of the event A, the following axioms are satisfied:

1. For every event A , $P(A) \geq 0$.
2. For a sample space S, $P(S) = 1$.
3. For any number of mutually exclusive events $A_1, A_2, \ldots,$

   $$P(\bigcup_i A_i) = \sum_i P(A_i).$$

   In particular, for two mutually exclusive events A_1 and A_2,

   $$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

4. If $A_1 \subseteq A_2$, then $P(A_1) \leq P(A_2)$ and $P(A_2 \setminus A_1) = P(A_2) - P(A_1)$.
5. For every event A, $0 \leq P(A) \leq 1$. This states that a probability is always between 0 and 1.
6. $P(\emptyset) = 0$. This indicates that the probability of an impossible event is zero.
7. If A^c is the complement of A, then $P(A^c) = 1 - P(A)$.
8. If $A = A_1 \cup A_2 \cup \ldots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then $P(A) = P(A_1) + P(A_2) + \ldots + P(A_n)$. In particular, if $A = S$ (the sample space), then $P(A_1) + P(A_2) + \ldots + P(A_n) = 1$.
9. If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. More generally, for three events A_1, A_2, A_3,

   $$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

   Generalizations to n events can also be made.

10. For any events A and B, $P(A) = P(A \cup B) + P(A \cap B)$.

11. If an event A must result in the occurrence of one of the mutually exclusive events $A_1, A_2, \ldots, A_n$, then $P(A) = P(A \cap A_1) + P(A \cap A_2) + \ldots + P(A \cap A_n)$.

Example 5. *Consider the experiment of tossing a single die. To find the probability of either a 2 or a 5 turning up:*

- *The sample space S is $\{1, 2, 3, 4, 5, 6\}$.*
- *Assuming the die is fair, each outcome has an equal probability of $\frac{1}{6}$.*
- *The event of getting either a 2 or a 5 can be denoted by $\{2, 5\}$.*
- *Therefore, the probability $P(\{2 \text{ or } 5\})$ is the sum of the probabilities of getting a 2 and getting a 5:*

$$P(\{2 \text{ or } 5\}) = P(\{2\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

0.2.5 Conditional Probability

Definition 3. *Conditional probability is the chance that an event will happen after another event has already occurred. Specifically, if we have two events, A and B, in a probability space, and the probability of A happening ($P(A)$) is greater than zero, then the conditional probability of B given A is calculated by the formula:*

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

This formula says that the probability of both A and B happening is equal to the probability of A happening times the probability of B happening, assuming A has already happened. We call $P(B \mid A)$ the conditional probability of B given A.

Example 6. *Let's calculate the probability of getting a number less than 4 on a single dice toss under two conditions:*

1. *Without any additional information.*

2. Knowing that the toss resulted in an odd number.

(a) *Define event B as the numbers less than 4, which includes 1, 2, and 3. Since each face of the die is equally likely, and there are three outcomes in event B out of six possible outcomes, the probability is:*

$$P(B) = \frac{3}{6} = \frac{1}{2}$$

(b) *Define event A as getting an odd number, which includes 1, 3, and 5. If we know the result is an odd number, we focus only on the numbers 1, 3, and 5. We need to find the probability of getting a number less than 4 among these. The outcomes 1 and 3 fit this criteria, so:*

$$P(B \mid A) = \frac{\textit{Number of favorable outcomes}}{\textit{Total outcomes in } A} = \frac{2}{3}$$

This analysis shows that knowing the result is an odd number increases the probability from $\frac{1}{2}$ to $\frac{2}{3}$.

0.2.6 Properties of Independent Events:

1. addresses the joint probability of three events, A_1, A_2, and A_3:

   - The probability that A_1, A_2, and A_3 all occur sequentially is calculated as:

     $$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2)$$

     This approach can be generalized to n events.

2. involves conditional probabilities among mutually exclusive events:

   - If an event A must result in one of several mutually exclusive events $A_1, A_2, \ldots, A_n$, then:

     $$P(A) = P(A_1) \times P(A \mid A_1) + P(A_2) \times P(A \mid A_2) + \ldots + P(A_n) \times P(A \mid A_n)$$

- If the occurrence of event B is not influenced by event A, i.e., $P(B \mid A) = P(B)$, then A and B are considered independent. This can be shown by:

$$P(A \cap B) = P(A) \times P(B)$$

Conversely, if the above equation holds, A and B are independent.

- For three events A_1, A_2, A_3 to be pairwise independent:

$$P(A_j \cap A_k) = P(A_j) \times P(A_k) \quad \text{for } j, k = 1, 2, 3$$

Additionally, for complete independence:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

Note that neither condition alone is sufficient for complete independence. Independence for more than three events follows similarly.

0.3 Bayes' Theorem or Rule

Definition 4. *Suppose $A_1, A_2, \ldots, A_n$ are mutually exclusive events that together cover the entire sample space S, meaning one of these events must occur. Given any event A, Bayes' Rule provides a way to compute the probabilities of the events $A_1, A_2, \ldots, A_n$ that could lead to the occurrence of A. This makes Bayes' theorem particularly valuable for determining the probability of causes. The theorem is stated as follows:*

$$P(A_i \mid A) = \frac{P(A \cap A_i)}{P(A)} = \frac{P(A_i)P(A \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(A \mid A_j)}$$

for each $i = 1, 2, \ldots, n$, where $P(A)$ is not zero.

Example 7. *A box contains 3 blue and 2 red marbles while another box contains 2 blue and 5 red marbles. A marble drawn at random from one of the boxes turns out to be blue. What is the probability that it came from the first box?*

Solution *We use Bayes' Theorem to solve this.* ***Define the Events:***

- *Let B be the event of drawing a blue marble.*
- *Let F_1 be the event of choosing from the first box.*
- *Let F_2 be the event of choosing from the second box.*

Assuming equal probability of choosing from either box:

$$P(F_1) = P(F_2) = 0.5$$

Probability of Drawing a Blue Marble:

$$P(B \mid F_1) = \frac{3}{5} \quad \textit{(3 blue marbles out of 5 total in the first box)}$$
$$P(B \mid F_2) = \frac{2}{7} \quad \textit{(2 blue marbles out of 7 total in the second box)}$$

Using the Law of Total Probability to Find $P(B)$***:***

$$P(B) = P(B \mid F_1)P(F_1) + P(B \mid F_2)P(F_2)$$
$$P(B) = \left(\frac{3}{5}\right)(0.5) + \left(\frac{2}{7}\right)(0.5)$$
$$P(B) = 0.3 + 0.142857 = 0.442857$$

Apply Bayes' Theorem to Find $P(F_1 \mid B)$***:***

$$P(F_1 \mid B) = \frac{P(B \mid F_1)P(F_1)}{P(B)}$$

$$P(F_1 \mid B) = \frac{\left(\frac{3}{5}\right) \times 0.5}{0.442857}$$
$$P(F_1 \mid B) = \frac{0.3}{0.442857} \approx 0.6778$$

Therefore, the probability that the blue marble came from the first box, given that a blue marble was drawn, is approximately 0.678 or 67.8%.

Example 8. *Each of three identical jewelry boxes has two drawers. The first box contains a gold watch in each drawer. The second box contains a silver watch in each drawer. The third box has one drawer with a gold watch and the other with a silver watch. If a box is selected at random, a drawer is opened, and it contains a silver watch, what is the probability that the other drawer contains a gold watch?*

Solution:

Define the events:

- *B_1, B_2, B_3 represent choosing the first, second, and third box, respectively.*
- *S represents observing a silver watch.*

Initial probabilities:

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

Conditional probabilities of observing a silver watch:

$$P(S \mid B_1) = 0, \quad P(S \mid B_2) = 1, \quad P(S \mid B_3) = \frac{1}{2}$$

Total probability of observing a silver watch:

$$P(S) = P(S \mid B_1)P(B_1) + P(S \mid B_2)P(B_2) + P(S \mid B_3)P(B_3)$$

$$P(S) = (0 \cdot \frac{1}{3}) + (1 \cdot \frac{1}{3}) + (\frac{1}{2} \cdot \frac{1}{3}) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Applying Bayes' Theorem to find $P(B_3 \mid S)$:

$$P(B_3 \mid S) = \frac{P(S \mid B_3)P(B_3)}{P(S)}$$

$$P(B_3 \mid S) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

Thus, the probability that the other drawer contains a gold watch, given that a silver watch was observed, is $\frac{1}{3}$.

Example 9. *Given three urns:*

- *Urn I has 2 white and 3 black balls.*
- *Urn II has 4 white and 1 black ball.*
- *Urn III has 3 white and 4 black balls.*

An urn is selected at random, and a ball is drawn that is found to be white. We wish to find the probability that Urn I was selected, using Bayes' Theorem. ***Solution:*** *Define the Events:*

- *W - drawing a white ball.*
- *U_1, U_2, U_3 - selecting Urn I, II, and III, respectively.*

Initial Probabilities:

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

Conditional Probabilities of Drawing a White Ball:

$$P(W \mid U_1) = \frac{2}{5}, \quad P(W \mid U_2) = \frac{4}{5}, \quad P(W \mid U_3) = \frac{3}{7}$$

Total Probability of Drawing a White Ball:

$$P(W) = P(W \mid U_1)P(U_1) + P(W \mid U_2)P(U_2) + P(W \mid U_3)P(U_3)$$

$$P(W) = \frac{2}{15} + \frac{4}{15} + \frac{3}{21}$$

$$P(W) = \frac{14}{35} + \frac{28}{35} + \frac{5}{35} = \frac{47}{105}$$

Applying Bayes' Theorem:

$$P(U_1 \mid W) = \frac{P(W \mid U_1)P(U_1)}{P(W)}$$

$$P(U_1 \mid W) = \frac{\frac{2}{15}}{\frac{47}{105}} = \frac{2}{15} \times \frac{105}{47} = \frac{14}{47}$$

Therefore, the probability that Urn I was selected given that a white ball was drawn is approximately $\frac{14}{47} \approx 0.2979$ *or about 29.79%.*

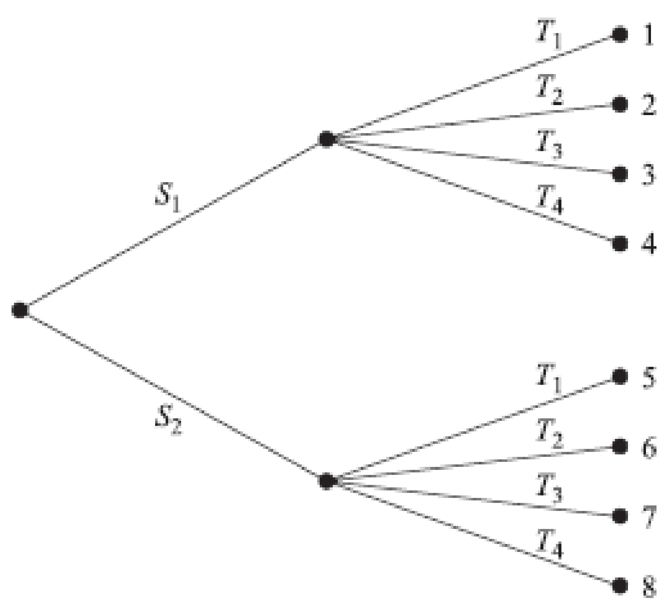
0.4 Combinatorial Analysis

In numerous instances, the total number of sample points within a sample space is relatively small, facilitating the direct enumeration or counting of these points to determine probabilities. Nonetheless, there are situations where direct counting is practically infeasible. Under these circumstances, combinatorial analysis is employed, which may be regarded as an advanced method of counting.

0.4.1 Tree Diagrams

If a task can be completed in n_1 ways, and after completing it, a second task can be done in n_2 ways, continuing in this manner until a k-th task which can be completed in n_k ways, then the total number of ways to complete all k tasks in the specified order is $n_1 \times n_2 \times \cdots \times n_k$.

Example 10. *If a man has 2 hats and 4 watches, then he can choose a hat and then a watch in* $2 \times 4 = 8$ *different ways. A diagram, known as a tree diagram due to its resemblance to a tree , is frequently utilized to illustrate this principle.*
Letting the hats be represented by S_1, S_2 *and the watch by* T_1, T_2, T_3, T_4*, the various ways of choosing a hat and then a watch are indicated in the tree diagram.*

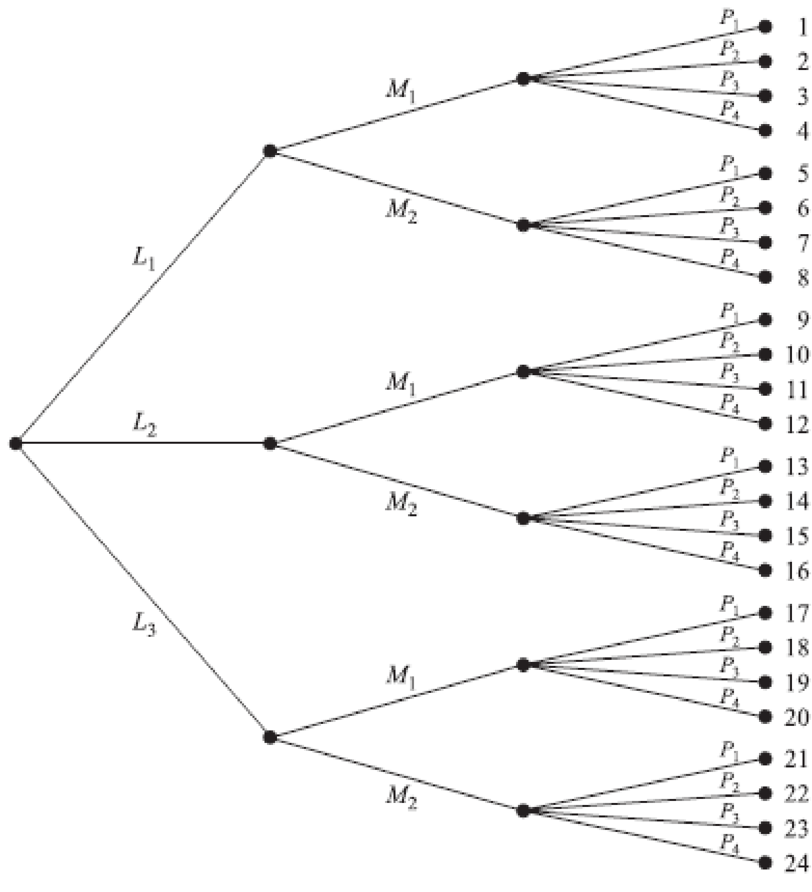


Example 11. *To form a committee of 3 members with representatives from labor, management, and the public, where there are 3 labor candidates (L), 2 management candidates (M), and 4 public candidates (P), we can calculate the number of possible committees. Applying the fundamental principle of counting, we select one representative from each group in sequence:*

- *Labor: 3 ways*
- *Management: 2 ways*
- *Public: 4 ways*

Thus, the total number of distinct committees is $3 \times 2 \times 4 = 24$.

Alternatively, by denoting labor as L_1, L_2, L_3*, management as* M_1, M_2*, and public as* P_1, P_2, P_3, P_4*, a tree diagram would illustrate the 24 unique committees that can be formed, such as* $L_1M_1P_1, L_1M_1P_2$*, etc.*



0.4.2 Permutations

When arranging r distinct objects from a collection of n distinct objects in a line, the fundamental principle of counting gives us n choices for the first position, $n-1$ for the second, and so on, until $n-r+1$ for the r-th position. Thus, the total number of arrangements, known as permutations, is:

$$P_n^r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-r+1)$$

This product has r factors, and P_n^r represents the permutations of n objects taken r at a time. Specifically, when $r = n$, we have:

$$P_n = n(n-1)(n-2)\dots 1 = n!$$

which defines n factorial. In the case of permutations with indistinguishable objects among n, with $n_1, n_2, \ldots, n_k$ being the count of each type and $n = n_1 + n_2 + \ldots + n_k$, the number of different permutations is given by a formula involving factorials. By definition, $0! = 1$.

Example 12. *The number of different arrangements, or permutations, consisting of 3 letters each that can be formed from the 7 letters A, B, C, D, E, F, G is given by*

$$P_7^3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$$

.

Remark 0.4.1. *Suppose that a set consists of n objects of which n_1 are of one type (i.e., indistinguishable from each other), n_2 are of a second type, ..., n_k are of a k-th type. Here, of course, $n = n_1 + n_2 + \cdots + n_k$. Then the number of different permutations of the objects is denoted as $P_n^{n_1, n_2, \ldots, n_k}$ and is given by:*

$$P_n^{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Example 13. *Five red marbles, two white marbles, and three blue marbles are arranged in a row. If marbles of the same color are not distinguishable from each other, we wish to find how many different arrangements are possible.*

Let N be the number of such arrangements. By multiplying N by the number of ways of arranging (a) the five red marbles among themselves, (b) the two white marbles among themselves, and (c) the three blue marbles among themselves—that is, by $5!$, $2!$, and $3!$ respectively—we equate this to the total number of arrangements for 10 distinct marbles, which is $10!$.

Thus, we have that $(5! \cdot 2! \cdot 3!)N = 10!$, and solving for N gives us:

$$N = \frac{10!}{5! \cdot 2! \cdot 3!}$$

In general, the number of different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_k are alike, is given by:

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

where $n_1 + n_2 + \ldots + n_k = n$.

Example 14. *The number of different permutations of the 11 letters of the word MISSISSIPPI, which consists of 1 M, 4 I's, 4 S's, and 2 P's, is given by the formula:*

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34,650$$

This equation calculates the total number of unique ways to arrange the letters by dividing the factorial of the total number of letters by the product of the factorials of the number of times each unique letter appears.

0.4.3 Combinations

In permutations, the order of objects is crucial; for instance, the sequence *abc* differs from *bca*. However, in many scenarios, the order of selection does not matter. These are termed as combinations, where *abc* and *bca* represent the same combination. The number of combinations for selecting r objects from n objects, often written as C_n^r, is calculated as:

$$C_n^r = \frac{n!}{r!(n-r)!}$$

This formula calculates how many different groups of r objects can be formed from n objects, disregarding the order of selection.

Example 15. *A box contains 8 red, 3 white, and 9 blue balls. If 3 balls are drawn at random without replacement, we want to determine the probability of the following events:*

(a) All 3 are red.

(b) All 3 are white.

(c) 2 are red and 1 is white.

(d) *At least 1 is white.*

(e) *1 of each color is drawn.*

(f) *The balls are drawn in the order red, white, blue.*

Calculations:

1. *For all 3 balls being red, the probability is:*

$$P(\textit{all red}) = \frac{C_8^3}{C_{20}^3} = \frac{56}{1140}$$

2. *For all 3 balls being white, the probability is:*

$$P(\textit{all white}) = \frac{C_3^3}{C_{20}^3} = \frac{1}{1140}$$

3. *For 2 red balls and 1 white ball, the probability is:*

$$P(2\ \textit{red}, 1\ \textit{white}) = \frac{C_8^2 C_3^1}{C_{20}^3} = \frac{84}{1140}$$

4. *For at least 1 white ball, the probability is:*

$$P(\textit{at least 1 white}) = 1 - P(\textit{no white}) = 1 - \frac{C_{17}^3}{C_{20}^3} = \frac{23}{57}$$

5. *For drawing one of each color, the probability is:*

$$P(1\ \textit{red}, 1\ \textit{white}, 1\ \textit{blue}) = \frac{C_8^1 C_3^1 C_9^1}{C_{20}^3} = \frac{216}{1140}$$

6. *For drawing the balls in the order red, white, blue, the probability is:*

$$P(\textit{red, white, blue}) = \frac{8}{20} \times \frac{3}{19} \times \frac{9}{18} = \frac{1}{19}$$