Correction of : Algebra tutorial series 3

Exercise 1 *Let the map $g : \mathbb{R} - \left\{\frac{1}{2}\right\} \to \mathbb{R}^*$ be such that :*

$$g(x) = \frac{9}{2x-1}$$

1. *Show that g is a bijection.*

a/ *g injective :*
$\forall x_1, x_2 \in \mathbb{R} - \left\{\frac{1}{2}\right\}, g(x_1) = g(x_2) \Rightarrow \frac{9}{2x_1-1} = \frac{9}{2x_2-1} \Rightarrow 9(2x_1-1) = 9(2x_2-1) \Rightarrow x_1 = x_2.$

b/ *g surjective :*
$\forall y \in \mathbb{R}^*, \exists x = \frac{9+y}{2y} \in \mathbb{R} - \left\{\frac{1}{2}\right\} \quad / y = g(x).$
We verify that $\frac{9+y}{2y} \neq \frac{1}{2}$:
Suppose that $\frac{9+y}{2y} = \frac{1}{2}$ then $9 + y = y$ which implies that $9 = 0$ contradiction, then $\frac{9+y}{2y} \neq \frac{1}{2}$.

$$\begin{aligned} g^{-1} : \ \mathbb{R}^* &\to \ \mathbb{R} \setminus \left\{\tfrac{1}{2}\right\} \\ x &\longmapsto \ g^{-1}(x) = \tfrac{9+x}{2x} \end{aligned}$$

2.
$$g^{-1}([-5, 2]) = g^{-1}([-5, 0[) \cup g^{-1}(\{0\}) \cup g^{-1}(]0, 2]).$$

$g^{-1}(\{0\}) = \{x \in \mathbb{R}^* / \quad g(x) = 0\} = \emptyset.$
We have : $(g^{-1})'(x) = \frac{-18}{4x^2} < 0$ which means that the function $g'$is decreasing, so :
For $-5 \leq x < 0$we have $g^{-1}(-5) = \frac{-2}{5} \geq g^{-1}(x) > \lim_{x \to 0_<} = -\infty$ then
$g^{-1}([-5, 0[) = \left]-\infty, \frac{-2}{5}\right].$
We also find $g^{-1}(]0, 2]) = \left[\frac{11}{4}, +\infty\right[$, with $g^{-1}(2) = \frac{11}{4}$.
We conclude that : $g^{-1}([-5, 2]) = \left]-\infty, \frac{-2}{5}\right] \cup \left[\frac{11}{4}, +\infty\right[$.

Exercise 2 ***(Exam January 2023)***
We define on $\mathbb{R}$, the composition law $*$ by : $\forall x, y \in \mathbb{R}, \quad x * y = x + y - 2.$

1. *Show that $(\mathbb{R}, *)$ is an abelian group.*

a/ $\forall x, y \in \mathbb{R}, \quad x * y = x + y - 2 \in \mathbb{R}$, *so $\mathbb{R}$ is closed under the operation $*$.*

b/ $\forall x, y \in \mathbb{R}, \quad x * y = x + y - 2 = y + x - 2 = y * x$, *so the law $*$ is commutative.*

c/ $\forall x, y, z \in \mathbb{R}, \quad x * (y * z) = x * (y + z - 2) = x + (y + z - 2) - 2 = (x + y - 2) + z - 2 = (x * y) * z$, *so the law $*$ is associative.*

d/ $\exists ? e \in \mathbb{R} / \quad \forall x \in \mathbb{R}, x * e = e * x = x.$
$x + e - 2 = x \Rightarrow e = 2.$

e/ $\forall x \in \mathbb{R}, \exists ? x^{-1} \in \mathbb{R} / \quad x * x^{-1} = x^{-1} * x = e = 2.$
$x * x^{-1} = x + x^{-1} - 2 = 2 \Rightarrow x^{-1} = 4 - x$
$(\mathbb{R}, *)$ is an abelian group.

2. *let $n \in \mathbb{N}^*$. We set $x^{(1)} = x$ and $x^{(n+1)} = x^{(n)} * x$.*

(a) $x^{(2)} = x^{(1)} * x = x * x = 2x - 2, \quad x^{(3)} = 3x - 4.$

(b) *Show that $\forall n \in \mathbb{N}^* : x^{(n)} = nx - 2(n-1)$.*

i/ *Base case : For $n = 1$we have : $x^{(1)} = 1x - 2(1-1) = x$.*

ii/ *Suppose that $x^{(n)} = nx - 2(n-1)$ and show that $x^{(n+1)} = (n+1)x - 2n$.*
We have : $x^{(n+1)} = x^{(n)} * x = x^{(n)} + x - 2 = nx - 2(n-1) + x - 2 = (n+1)x - 2n$.

3. *Let $A = \{x \in \mathbb{R} : x \quad is \quad even\}$. Show that $(A, *)$ is a subgroup of$(\mathbb{R}, *)$.*

a/ $e = 2 \in A.$

b/ $\forall x \in A\,(x = 2p), \quad \forall y \in A\,(y = 2q), \quad x * y = 2p * 2q = 2p + 2q - 2 = 2(p + q - 2) \in A.$

c/ $\forall x = 2p \in A, x^{-1} = 4 - x = 4 - 2p = 2(2 - p) \in A.$

Exercise 3 *Let $(G, .)$ be a group, we denote by $Z(G) = \{x \in G / \quad \forall y \in G, xy = yx\}$ **the center** of G.*

1. *Show that $Z(G)$ is a subgroup of G.*

i/ $e \in Z(G)$ *because* $ey = ye = y, \forall y \in G.$

ii/ $\forall x, x' \in G, x * x' \in ?G.$
We have $x \in G \Leftrightarrow \forall y \in G, xy = yx$ *and* $x' \in G \Leftrightarrow \forall y \in G, x'y = yx'.$
Then $\forall y \in G, (xx')y = x(x'y) = x(yx') = (xy)x' = (yx)x' = y(xx')$ *which implies that* $xx' \in G.$

iii/ $\forall x \in G, x^{-1} \in G,$ *we must have* $\forall y \in G, x^{-1}y = yx^{-1}.$
So $\forall y \in G, x^{-1}y = (y^{-1}x)^{-1} = (xy^{-1})^{-1} = yx^{-1}.$

Note : $xy = x.y$**,** $(xy)^{-1} = y^{-1}x^{-1}$**,** e ***the identity element and*** x^{-1} ***the inverse of*** x**.**

Exercise 4 *Let $*$ be a binary operation on $\mathbb{R}^2$ defined by :*

$$\forall (x, y), (x', y') \in \mathbb{R}^2, \quad (x, y) * (x', y') = (x + x', y + y' + 2xx')$$

1. *Show that $(\mathbb{R}^2, *)$ is an abelian group.*

a/ $\forall (x, y), (x', y') \in \mathbb{R}^2, \quad (x, y) * (x', y') = (x + x', y + y' + 2xx') \in \mathbb{R}^2,$ *so* $\mathbb{R}^2$ *is closed under the operation* $*$.

b/ $\forall (x, y), (x', y') \in \mathbb{R}^2, \quad (x, y)*(x', y') = (x+x', y+y'+2xx') = (x'+x, y'+y+2x'x) = (x', y')*(x, y)$ *so the law* $*$ *is commutative.*

c/ $\forall (x, y), (x', y'), (x", y") \in \mathbb{R}^2,$
$((x, y) * (x', y')) * (x", y") = ((x + x', y + y' + 2xx')) * (x", y") =$
$(x + x' + x", y + y' + 2xx' + y" + 2(x + x')x") \cdots\cdots (1)$
$(x, y) * ((x', y') * (x", y")) = (x, y) * (x' + x", y' + y" + 2x'x") =$
$(x + x' + x", y + (y' + y" + 2x'x") + 2x(x' + x")) \cdots\cdots (2).$
$(1) = (2)$, *so the law* $*$ *is associative.*

d/ $\exists ?(e_1, e_2) \in \mathbb{R}^2 / \quad \forall (x, y) \in \mathbb{R}^2, (x, y) * (e_1, e_2) = (e_1, e_2) * (x, y) = (x, y).$
We have $(x, y) * (e_1, e_2) = (x + e_1, y + e_2 + 2xe_1) = (x, y)$ *wich gives :*
$x + e_1 = x \Rightarrow e_1 = 0$ *and* $y + e_2 + 2x0 = y \Rightarrow e_2 = 0.$
Then $(e_1, e_2) = (0, 0).$

d/ $\forall (x, y) \in \mathbb{R}^2, \exists ?(x^{-1}, y^{-1}) \in \mathbb{R}^2 / \quad (x, y) * (x^{-1}, y^{-1}) = (x^{-1}, y^{-1}) * (x, y) = (0, 0).$
$(x, y) * (x^{-1}, y^{-1}) = (x + x^{-1}, y + y^{-1} + 2xx^{-1}) = (0, 0) \Rightarrow x + x^{-1} = 0, i.e. \quad x^{-1} = -x$
and $y + y^{-1} + 2xx^{-1} = 0 \Rightarrow y^{-1} = -y - 2x(-x) = 2x^2 - y.$ *So* $(x^{-1}, y^{-1}) = (-x, 2x^2 - y).$

2. *Show that the curve of equation $y = x^2$ is a subgroup of $(\mathbb{R}^2, *)$ which we will denote P.*
i.e. $P = \{(x, x^2) / \quad x \in \mathbb{R}\}$ *is a subgroup of* $(\mathbb{R}^2, *)$.

i/ $(0, 0) = (0, 0^2) \in P.$

ii/ $\forall (x, y), (x', y') \in P(y = x^2, y' = x'^2),$
we have $(x, y) * (x', y') = (x, x^2) * (x', x'^2) = (x + x', x^2 + x'^2 + 2xx') = (x + x', (x + x')^2) \in P.$

iii/ $\forall (x, y) \in P, (y = x^2), (x^{-1}, y^{-1}) = (-x, 2x^2 - y) = (-x, 2x^2 - x^2) = (-x, (-x)^2) \in P.$

3. *Show that the map $\phi : (\mathbb{R}, +) \to (P, *)$, defined by $\phi(x) = (x, x^2)$ is a group isomorphism.*
   (a) $\forall x, x' \in \mathbb{R}, \phi(x + x') = (x + x', (x + x')^2) = (x + x', x^2 + x'^2 + 2xx') = (x, x^2) * (x', x'^2)$
   (b) ϕ *bijective.*

Exercise 5

$$\forall a, b \in A, \quad a \oplus b = a + b + 1, \qquad a \otimes b = a \cdot b + a + b$$

$f : (A, +, \cdot) \longrightarrow (A, \oplus, \otimes)$, *given by* $f(a) = a - 1$
f *an isomorphisme of rings i.e.* f *an homomorphism of rings and* f *is bijective.*

1. $\forall a, b \in A, \quad f(a+b) = a+b-1, \quad$ *and* $\quad f(a) \oplus f(b) = (a-1) \oplus (b-1) = (a-1)+(b-1)+1.$
   So $f(a+b) = f(a) \oplus f(b)$ *(group homomorphism).*
2. *Similarly, we find* $\forall a, b \in A, \quad f(a \cdot b) = f(a) \otimes f(b).$
3. f *bijective* $\forall b \in A, \exists!?(unique) a \in A/ \quad b = f(a) = a-1.$
   Indeed : $\forall b \in A, \exists! a = b+1 \in A/ \quad b = f(a).$

Exercise 6 $\mathbb{Z}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2}/ \quad a, b \in \mathbb{Z}\right\}$ *subring of* $\mathbb{R}$.

1.

i/ $0 + 0\sqrt{2} \in \mathbb{Z}\left[\sqrt{2}\right]$. *So* $\mathbb{Z}\left[\sqrt{2}\right] \neq \emptyset$.

ii/ $\forall x \in \mathbb{Z}\left[\sqrt{2}\right] (x = a + b\sqrt{2}, \quad a, b \in \mathbb{Z}), \qquad \forall x' \in \mathbb{Z}\left[\sqrt{2}\right] (x' = a' + b'\sqrt{2}, \quad a', b' \in \mathbb{Z}),$
$x + x' = (a + a') + (b + b')\sqrt{2} \in \mathbb{Z}\left[\sqrt{2}\right]$ $(a + a' \in \mathbb{Z}$ *and* $b + b' \in \mathbb{Z})$.
Then $x + x' \in \mathbb{Z}\left[\sqrt{2}\right]$.

iii/ $\forall x \in \mathbb{Z}\left[\sqrt{2}\right] (x = a + b\sqrt{2}, \quad a, b \in \mathbb{Z})$ *we have* $-x = -(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} \in \mathbb{Z}\left[\sqrt{2}\right]$,
$((-a), (-b) \in \mathbb{Z})$.
$\mathbb{Z}\left[\sqrt{2}\right]$ *is a subgroup of* $(\mathbb{R}, +)$.

2. $\forall x \in \mathbb{Z}\left[\sqrt{2}\right] (x = a + b\sqrt{2}, a, b \in \mathbb{Z}), \qquad \forall x' \in \mathbb{Z}\left[\sqrt{2}\right] (x' = a' + b'\sqrt{2}, a', b' \in \mathbb{Z}),$
   $x \cdot x' = (a + b\sqrt{2}) \cdot (a' + b'\sqrt{2}) = (aa' + 2bb') + (ab' + ba')\sqrt{2} \in \mathbb{Z}\left[\sqrt{2}\right], \quad ((aa' + 2bb') \in \mathbb{Z})$ *and* $((ab' + ba') \in \mathbb{Z})$,
   then $x \cdot x' \in \mathbb{Z}\left[\sqrt{2}\right]$.
   $\mathbb{Z}\left[\sqrt{2}\right]$ *is a subring of* $(\mathbb{R}, +, \cdot)$.