Course : Algebra 3
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Department of Computer Science

Chapter 3 :

Endomorphisms

1 Properties of eigenvalues and of eigenvectors

Definition 1.1 *Suppose that A is an $n \times n$ matrix and that*

$$AX = \lambda X, \tag{1}$$

is a linear system where X is a nonzero vector. Then, λ is a so-called eigenvalue of the matrix A.

Definition 1.2 *Let us assume that*

$$AX = \lambda X, \tag{2}$$

is a linear system where A is an $n \times n$ matrix, X is a nonzero vector, and where λ is an eigenvalue. Then, X is called an eigenvector of A.

Definition 1.3 *Let A be an $n \times n$ matrix and let*

$$D = P^{-1}AP. \tag{3}$$

Then, we can say that D is similar to the matrix A if and only if the $n \times n$ matrix P is nonsingular and satisfies (3).

Theorem 1.1 *Let A and D be similar such that*

$$A = PDP^{-1}. \tag{4}$$

Then in the case where $D(P^{-1})X = \lambda(P^{-1}X)$, we have

$$AX = \lambda X.$$

Corollary 1.1 *Let λ be nonzero. Then, we find that the eigenvalues of the product A^TA, denoted by λ, are the eigenvalues of AA^T.*

Theorem 1.2 *Suppose that* $p(\lambda)$ *satisfies*

$$p(\lambda) = |A - \lambda I| = \prod_{i=1}^{n} (\lambda_i - \lambda).$$

Then, we have

$$tr(A) = \sum_{i=1}^{n} \lambda_i,$$

and

$$|A| = p(0) = \prod_{i=1}^{n} \lambda_i.$$

Theorem 1.3 *Suppose that* A *is an* $n \times m$ *matrix and that* B *is an* $m \times n$ *matrix. Then, we have*

$$(-\lambda)^{n-m} |BA - \lambda I_m| = |AB - \lambda I_n|, \; for \; m \preccurlyeq n. \tag{5}$$

Theorem 1.4 *Let* A *be a diagonal matrix, an upper triangular matrix, or a lower triangular matrix. Then,* λ *must be the diagonal entries of this matrix.*

Theorem 1.5 *Let* A *and* B *be* $n \times n$ *matrices. Suppose that* A *and* B *are similar matrices. Then, we have that the eigenvalue of* A *is equal to the eigenvalue of* B.

2 Characteristic polynomials

Definition 2.1 *Let* A *be a square matrix and suppose that* $p(\lambda)$ *is the so-called characteristic polynomial of* A *where*

$$p(\lambda) = \det(A - \lambda I).$$

Then, the roots of

$$p(\lambda) = 0$$

represent the eigenvalues of A.

Theorem 2.1 *Assume that*

$$p(\lambda) = \det(A - \lambda I),$$

where A *is an* $n \times n$ *matrix. Then, we have*

$$p(A) = 0.$$

Theorem 2.2 *Suppose that A and B are square and similar matrices. Then, the characteristic polynomial of A is equal to the characteristic polynomial of B, i.e.*

$$\det(A - \lambda I) = \det(B - \lambda I).$$

Theorem 2.3 *Let A be a square matrix and let A^T denote the transpose of A. Then, the characteristic polynomial of A is equal to the characteristic polynomial of A^T, i.e.*

$$\det(A - \lambda I) = \det(A^T - \lambda I).$$

3 Diagonalizable matrices

Definition 3.1 *Suppose that U and V are K-vector spaces and that $f : U \longrightarrow V$ satisfies, for $\alpha, \beta \in K$,*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \; for \; x, y \in U.$$

Then, f is said to be a homomorphism, in other words, a linear transformation.

Definition 3.2 *Suppose that U is a K-vector space. Then, a K-endomorphism represents a K-linear transformation $U \longrightarrow U$.*

Theorem 3.1 *Suppose that A is a matrix and that an endomorphism $f : U \longrightarrow U$ is associated with A. Then, A is said to be diagonalizable if and only if the eigenvectors $X_1, X_2, \cdots, X_n$ must be linearly independent.*

Remark 3.1 *.*
Let A be an $n \times n$ matrix and let D denote a diagonal matrix. In the case where there exists P with P is a square and nonsingular matrix such that

$$D = P^{-1}AP. \tag{6}$$

Then, A is said to be diagonalizable.

Theorem 3.2 *Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A. Under the condition that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, then we can say that the matrix A is diagonalizable.*

4 Systems of differential equations

A system of differential equations is given by

$$X^{'} = AX, \tag{7}$$

where A is a coefficient matrix defined as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

and where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} and\ X^{'} = \begin{pmatrix} x_1^{'} \\ x_2^{'} \\ \vdots \\ x_n^{'} \end{pmatrix}.$$

Under the condition that A is diagonalizable, it is possible to define a matrix P. Here, the product $P^{-1}AP$ leads to get a diagonal matrix which is denoted by D, i.e.

$$D = P^{-1}AP. \tag{8}$$

Moreover, the homogeneous system of differential equations

$$X^{'} = AX, \tag{9}$$

can be solved by taking

$$X = PY.$$

Thus, we have

$$PY^{'} = APY,$$

which yields

$$Y^{'} = P^{-1}APY,$$

it follows from $D = P^{-1}AP$ that

$$Y^{'} = DY.$$

This means, for $i = 1, \cdots, n$,

$$y_i^{'} = \lambda_i y_i. \tag{10}$$

Then, the solution of (10) is given by

$$y_i = C_i \exp(\lambda_i t),\ for\ i = 1, \cdots, n.$$

The solution of (7) is $X = PY$ where

$$Y = \begin{pmatrix} C_1 \exp(\lambda_1 t) \\ C_2 \exp(\lambda_2 t) \\ \vdots \\ C_n \exp(\lambda_n t) \end{pmatrix}.$$

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