Algèbre 02

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1 Vector Spaces

When you read the word vector you may immediately think of two points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) connected by an arrow. Mathematically speaking, a vector is just an element of a vector space. This then begs the question : What is a vector space ? Roughly speaking, a vector space is a set of objects that can be added and multiplied by scalars.

Definition 1.0.1 *A vector space is a set E of objects, called vectors, on which two operations called addition and scalar multiplication have been defined satisfying the following properties. If u, v, w are in E and if $\alpha, \beta \in \mathbb{R}$ are scalars :*

1. *The sum $u + v$ is in E. (**closure under addition**)*
2. *$u + v = v + u$ (addition is commutative)*
3. *$(u + v) + w = u + (v + w)$ (addition is associative)*
4. *There is a vector in E called the zero vector, denoted by 0, satisfying $v + 0 = v$.*
5. *For each v there is a vector $-v$ in E such that $v + (-v) = 0$.*
6. *The scalar multiple of v by α, denoted $\alpha \cdot v$, is in E. (**closure under scalar multiplication**)*
7. *$\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.*
8. *$(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.*
9. *$(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$.*
10. *$1 \cdot v = v$*

Remark 1.0.1

1. *Elements of E are called vectors, and elements of $\mathbb{R}$ are called scalars. Instead of vector space on $\mathbb{R}$ we also say, $\mathbb{R}-$ vector space.*
2. *It can be shown that $0 \cdot v = 0$ for any vector v in E.*

To better understand the definition of a vector space, we first consider a few elementary examples.

Example 1.0.1

1. *$\mathbb{R}^2, \mathbb{R}^3$ and more generally $\mathbb{R}^n$ are real vector spaces.*
2. *The set of applications from $\mathbb{R}$ into $\mathbb{R}$ is a vector space on $\mathbb{R}$.*

3. *Let E be the unit disc in $\mathbb{R}^2$:*

$$E = \left\{(x, y) \in \mathbb{R}^2 / \quad x^2 + y^2 \leq 1\right\}$$

The circle is not closed under scalar multiplication. For example, take $u = (1, 0) \in E$ and multiply by say $\alpha = 2$. Then $\alpha u = (2, 0)$ is not in E. Therefore, property (6) of the definition of a vector space fails, and consequently the unit disc is not a vector space.

4. *Let E be the graph of the quadratic function $f(x) = x^2$:*

$$E = \left\{(x, y) \in \mathbb{R}^2 / \quad y = x^2\right\}$$

The set E is not closed under scalar multiplication. For example, $u = (1, 1)$ is a point in E but $2u = (2, 2)$ is not. You may also notice that E is not closed under addition either. For example, both $u = (1, 1)$ and $v = (2, 4)$ are in E but $u + v = (3, 5)$ and $(3, 5)$ is not a point on the parabola E. Therefore, the graph of $f(x) = x^2$ is not a vector space.

5. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ ***: The vector space of functions from*** $\mathbb{R}$ ***into*** $\mathbb{R}$.

a/ *Let f and g two elements of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. The function $f + g$ is defined by :*

$$\forall x \in \mathbb{R}, \quad (f + g)(x) = f(x) + g(x)$$

b/ *If λ is a real number and f is a function of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the function $\lambda.f$ is defined by the image of any real x as follows :*

$$\forall x \in \mathbb{R}, \quad (\lambda.f)(x) = \lambda f(x)$$

c/ ***The identity*** *The identity for addition is the null function, defined by :*

$$\forall x \in \mathbb{R}, \quad f(x) = 0.$$

This function can be written $0_E = 0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$.

d/ ***The inverses*** *The inverse of f in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is the function g from $\mathbb{R}$ to $\mathbb{R}$ defined by :*

$$\forall x \in \mathbb{R}, \quad g(x) = -f(x).$$

The inverse of f is noted $-f$.

6. *Let $E = \mathbb{R}_2[X] = \{P = aX^2 + bX + c, \quad a, b, c \in \mathbb{R})$ be the set of polynomials of degree less than or equal to 2, with coefficients in $\mathbb{R}$, provided with the following operations :*

a/ *A law " + ", given by : $\forall P, Q \in E, \quad P = aX^2 + bX + c, \quad Q = a'X^2 + b'X + c'$,*

$$P + Q = (a + a')X^2 + (b + b')X + (c + c').$$

b/ *A law " $\cdot$ " defined by : $\forall \alpha \in \mathbb{R}, \quad \forall P \in E, \quad P = aX^2 + bX + c$,*

$$\alpha \cdot P = (\alpha a)X^2 + (\alpha b)X + (\alpha c).$$

$(E, +, \cdot)$ is a vectorial space on $\mathbb{R}$.

1.1 Subspaces of Vector Spaces

Frequently, one encounters a vector space F that is a subset of a larger vector space E. In this case, we would say that F is a subspace of E. Below is the formal definition.

Definition 1.1.1 *Let E be a vector space. A subset F of E is called a subspace of E if it satisfies the following properties :*

1. *The zero vector of E is also in F.*
2. *F is closed under addition, that is, if u and v are in F then $u+v$ is in F.*
3. *F is closed under scalar multiplication, that is, if u is in $F$and α is a scalar then $\alpha \cdot u$ is in F.*

Example 1.1.1 *Let F be the graph of the function $f(x) = 2x$:*

$$F = \left\{(x, y) \in \mathbb{R}^2 | y = 2x\right\}.$$

F a subspace of $E = \mathbb{R}^2$.
If $x = 0$ then $y = 2 \cdot 0 = 0$ and therefore $(0, 0)$ is in F.
Let $u = (a, 2a)$ and $v = (b, 2b)$ be elements of F. Then $u+v = (a, 2a)+(b, 2b) = (a+b, 2a+2b) = (a + b, 2(a + b))$ Because the x and y components of $u + v$ satisfy $y = 2x$ then $u + v$ is inside in F. Thus, F is closed under addition.
Let α be any scalar and let $u = (a, 2a)$ be an element of F. Then $\alpha u = (\alpha a, \alpha 2a) = (\alpha a, 2\alpha a) \in F$. F is closed under scalar multiplication.
All three conditions of a subspace are satisfied for F and therefore F is a subspace of E.

Example 1.1.2 *Let F be the first quadrant in $\mathbb{R}^2$:*

$$F = \left\{(x, y) \in \mathbb{R}^2 | \quad x \geq 0, y \geq 0\right\}.$$

The set F contains the zero vector and the sum of two vectors in F is again in F. However, F is not closed under scalar multiplication. For example if $u = (1, 1)$ and $\alpha = -1$, then $\alpha u = (-1, -1)$ is not in F because the components of αu are clearly not non-negative.

Example 1.1.3 *Let $E = \mathbb{R}_n[t]$ and consider the subset F of E :*

$$F = \{P(t) \in \mathbb{R}_n[t] / \quad P'(1) = 0\}$$

*F **is a subspace of** E.*
The zero polynomial $0(t)$ clearly has derivative at $t = 1$ equal to zero, that is, $0'(1) = 0$, and thus the zero polynomial is in F. Now suppose that $P(t)$ and $Q(t)$ are two polynomials in F. Then, $P'(1) = 0$ and also $Q'(1) = 0$, from the rules of differentiation, we compute $(P + Q)'(1) = P'(1) + Q'(1) = 0 + 0$.
Therefore, the polynomial $(P + Q)(t)$ is in F, and thus F is closed under addition.
Now let α be any scalar and let $P(t)$ be a polynomial in F. Then $P'(1) = 0$. Using the rules of differentiation, we compute that $(\alpha P)'(1) = \alpha P'(1) = \alpha.0 = 0$. Therefore, the polynomial $(\alpha P)(t)$ is in F and thus F is closed under scalar multiplication.
All three properties of a subspace hold for F and therefore F is a subspace of $\mathbb{R}_n[t]$.

Example 1.1.4 1. *Any field $\mathbb{K}$ is a vectorspace on $\mathbb{K}$.*

2. *Any field $\mathbb{L}$ containing a field $\mathbb{K}$ is a vector space on $\mathbb{K}$ and $\mathbb{K}$ is a vector subspace of $\mathbb{L}$.*
3. *$\mathbb{C}$ is a vector space on $\mathbb{R}$ and $\mathbb{R}$ is a subspace of $\mathbb{C}$.*

Example 1.1.5 *Consider $F = \{(x,y) \in \mathbb{R}^2 / \quad x^2 + y^2 < 0\}$, $F = \emptyset$, so F is not a subspace of $\mathbb{R}^2$.*

Example 1.1.6 *Let $F = \{(x,y) \in \mathbb{R}^2 / \quad x - y + 1 = 0\}$, we have : $0_{\mathbb{R}^2} = (0,0) \notin F$, since $0 - 0 + 1 \neq 0$ therefore F is not a subspace of $\mathbb{R}^2$.*

Example 1.1.7 *Let $F = \{(x,y) \in \mathbb{R}^2 / \quad xy \geq 0\}$, we have $(2,1), (-1,-2) \in F$, but $(2,1) + (-1,-2) = (1,-1) \notin F$because does not check $xy \geq 0$ so F is not a subspace of $\mathbb{R}^2$.*

1.2 Operation on vector subspaces

Proposition 1.2.1 *Let $\mathbb{K}$ be a field, E a $\mathbb{K}-$vector space, F and G two subspaces of E, then :*

1. *$F \cap G$ is a subspace of E.*
2. *$F \cup G$ is a subspace of E if and only if, $F \subset G$or $G \subset F$.*

Proof 1.2.1 *(of 1.) We have F and G are subspaces of E, then : ($F \subset E$ and $G \subset E$ therefore $F \cap G \subset E$.)*

a/ $0_E \in F$ and $0_E \in G$ which means that $0_E \in F \cap G$.

b/ $\forall \alpha, \beta \in \mathbb{K}, \quad \forall x, y \in F \cap G$ (i.e. $x \in F \wedge x \in G$), we have $\alpha x + \beta y \in F$ and $\alpha x + \beta y \in G$, therefore $\alpha x + \beta y \in F \cap G$. Then $F \cap G$ is a subspace of E.

Remark 1.2.1 *We generalize the property (1) to any family of vector subspaces, i.e. If $(F_i)_{i \in I, I \subset \mathbb{N}}$, is a family of subvector spaces, then $\cap_{i \in I} F_i$ is a subspace.*

Example 1.2.1 *Let $E = \mathbb{R}^2$ be the vector space on $\mathbb{R}$. Consider the following subspaces F and G :*

$$F = \left\{(x,y) \in \mathbb{R}^2 / \quad y = 0\right\}, \qquad G = \left\{(x,y) \in \mathbb{R}^2 / \quad x = 0\right\}.$$

F and G are the x-axis and y-axis respectively.
Since $(1,0) \in F$ with $(1,0) \notin G$, then $F \nsubseteq G$ and $(0,1) \in G$ with $(0,1) \notin F$, then $G \nsubseteq F$. Therefore, $F \cup G$ is not a subspace of $\mathbb{R}^2$.
The result can be obtained by noting that $(1,0), (0,1) \in F \cup G$ but $(1,0) + (0,1) = (1,1) \notin F$ and $(1,1) \notin G$ then $(1,1) \notin F \cup G$. This means that $F \cup G$ is not a subspace of E.

Theoreme 1.2.1 *Let $\mathbb{K}$ be a field, E a vector space on $\mathbb{K}$, F and G two subspaces of E.*
The set $F + G$ defined by

$$F + G = \{x + y / \quad x \in F \quad \text{and} \quad y \in G\} \subset E$$

is a subspace of E called ***sum*** *of the subspaces F and G. If in addition $F \cap G = \{0_E\}$, we say that the sum $F + G$ is a* ***direct sum*** *and we write $F \oplus G$.*

Proof 1.2.2 *$F + G$ is a subspace of E :*

1. *$0_E = 0_E + 0_E \in F + G$ because $0_E \in F$ and $0_E \in G$ since F and G are two subspaces of E.*

2. $\forall \alpha, \beta \in \mathbb{K}, \quad \forall z, z' \in F+G$, *then* $z = x+y$ *and* $z' = x'+y'$*with* $x, x' \in F$ *and* $y, y' \in G$. *Since* F *and* G *are subspaces of* E, *then*

$$\alpha x + \beta x' \in F \quad \text{and} \quad \alpha y + \beta y' \in G.$$

This means that $(\alpha x + \beta x') + (\alpha y + \beta y') \in F + G$.

Therefore $(\alpha x+\beta x')+(\alpha y+\beta y') = \alpha(x+y)+\beta(x'+y') \in F+G$, *i.e.* $\alpha z + \beta z' \in F+G$

Example 1.2.2 *Consider the vector space* $\mathbb{R}^3$, *the subspaces* F *and* H *given by*

$$F = \left\{(x,y,z) \in \mathbb{R}^3 / x+y-z=0\right\} \quad \text{and} \quad H = \left\{(x,y,z) \in \mathbb{R}^3 / x=y=0\right\}.$$

We have $F + G = F \oplus G$. *Indeed :*
Let $(x,y,z) \in F \cap H$, *so* $(x,y,z) \in F$, *i.e.* $z = x+y$ *and* $(x,y,z) \in H$ *i.e.* $x = y = 0$, *so* $x = y = z = 0$, *therefore* $F \cap H = \{0_{\mathbb{R}^3}\}$.

Example 1.2.3 *For any vector space* E, *there are two trivial subspaces in* E, *namely,* E *itself is a subspace of* E *and the set consisting of the zero vector* $F = \{0\}$*is a subspace of* E.

There is one particular way to generate a subspace of any given vector space E *using the span of a set of vectors.*

2 Linear combinations, generating famillies, linearly independant famillies, bases, dimension.

2.1 Linear combinations

Let $v_1, v_2, \cdots, v_n$ *be a familly of vectors of a vector space on* $\mathbb{K}$, *We call linear combination of these vectors any vector of type*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

The scalars $\lambda_1, \cdots, \lambda_n$ *are called the coefficients of the linear combination.*
The span *of* $\{v_1, v_2, \cdots, v_n\}$ *is the set of all linear combinations of* $v_1, v_2, \cdots, v_n$.

$$span\,\{v_1, v_2, \cdots, v_n\} = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n / \quad \lambda_1, \cdots, \lambda_n \in \mathbb{R}\}$$

The span of a set of vectors in E *is a subspace of* E.

2.2 Generating famillies

Definition 2.2.1 *The family* $\{v_1, v_2, \cdots, v_n\}$ *is a generating family of the vector space* E *if every vector of* E *is a linear combination of the vectors* $v_1, v_2, \cdots, v_n$. *This can also be written :*

$$\forall v \in E, \quad \exists \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{K} / \quad v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

We also say that the family $\{v_1, v_2, \cdots, v_n\}$ *generates the vector space* E *and we write*

$$E = span\,\{v_1, v_2, \cdots v_n\}$$

.

Example 2.2.1 *Let the vectors* $v_1 = (2,1), v_2 = (1,1) \in \mathbb{R}^2$ *The vectors* $\{v_1, v_2\}$ *form a generating family of* $\mathbb{R}^2$*. Indeed, let* $v = (x,y) \in \mathbb{R}^2$*, showing that* v *is a linear combination of* v_1 *and* v_2 *is equivalent to demonstrate the existence of two real numbers* α *and* β *such that* $v = \alpha v_1 + \beta v_2$*. So we need to study the existence of solutions to the system :*

$$\begin{aligned} 2\alpha + \beta &= x \\ \alpha + \beta &= y \end{aligned}$$

Its solutions are $\alpha = x - y$ *and* $\beta = -x + 2y$*, whatever the real numbers* x *and* y*.*
This proves that there can be several different finite families, not included in each other, generating the same vector space.

Example 2.2.2 *Let* $E = \mathbb{R}_n[X]$ *be the vector space of polynomials of degree* $\leq n$*. Then the polynomials* $\{1, X, \cdots, X^n\}$ *form a generating family of* E*.*

2.3 Linearly independent famillies

Definition 2.3.1 *1. A familly* $\{v_1, v_2, \cdots, v_n\}$ *of vectors of a vector space* E *is linearly independent if the only linear combination of these vectors equal to the zero vector is the one whose coefficients are all zero. We also say that vectors* $\{v_1, v_2, \cdots, v_n\}$ *are linearly independent.*
This can be expressed as :
$\{v_1, v_2, \cdots, v_n\}$ *is a linearly independent familly is equivalent to :*

$$((\lambda_1, \cdots, \lambda_n) \in \mathbb{K}^n and \quad \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_p = 0_E) \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

2.4 Linearly dependent famillies

Definition 2.4.1 *1. A non linearly independent familly is called a linearly dependent familly. We also say that vectors* $\{v_1, v_2, \cdots, v_n\}$ *are linearly dependents.*
This can be expressed as : $\{v_1, v_2, \cdots, v_n\}$ *is a linearly dependent familly is equivalent to*

$$(\exists (\lambda_1, \cdots, \lambda_n) \in \mathbb{K}^n - \{0_{\mathbb{K}^n}\} / \quad \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_p = 0_E).$$

Example 2.4.1 *The polynomials* $P_1(X) = 1 - X, P_2(X) = 5 + 3X - 2X^2$ *and* $P_3(X) = 1 + 3X - X^2$ *form a linearly dependent family in the vector space* $\mathbb{R}_2[X]$*, because* $3P_1(X) - P_2(X) + 2P_3(X) = 0$*.*

Example 2.4.2 *In the vector space* $\mathcal{F}(\mathbb{R}, \mathbb{R})$ *of functions from* $\mathbb{R}$ *into* $\mathbb{R}$*, consider the family* $\{\cos, \sin\}$*. Let's show that it's a linearly independent family.*
Suppose we have $\lambda \cos + \mu \sin = 0$*, which is equivalent to* $\forall x \in \mathbb{R}, \lambda \cos(x) + \mu \sin(x) = 0$*. In particular, for* $x = 0$*, this equality gives* $\lambda = 0$*. And for* $x = \pi/2$*, it gives* $\mu = 0$*. So* $\{\cos, \sin\}$ *is a linearly independent family.*
On the other hand, the family $\left\{\cos^2, \sin^2 1\right\}$ *is linearly dependent because we have :* $\cos^2 + \sin^2 - 1 = 0$*.*
The coefficients of the linear dependence are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$*.*

Example 2.4.3 *In the vector space* $\mathbb{R}^4$ *defined over the field* $\mathbb{R}$*, consider the following vectors :*

$$v_1 = (1, 0, -1, 1), v_2 = (0, 1, 1, 0), v_3 = (1, 0, 0, 1), v_4 = (0, 0, 0, 1), v_5 = (1, 1, 0, 1).$$

The set $\{v_1, v_2, v_3, v_4\}$ *is linearly independent (to be verified). The set* $S_2 = \{v_1, v_2, v_5\}$ *is linearly dependent* $(v_5 = v_1 + v_2)$*.*

Theoreme 2.4.1 *Let E be a vector space over the field $\mathbb{K}$. A set $F = \{v_1, v_2, \cdots, v_n\}$ of n vectors of E, $(n > 2)$ is linearly dependent if and only if at least one of the vectors of F is a linear combination of the other vectors of F.*

Remark 2.4.1

1. *Any family containing a linearly dependent family is linearly dependent.*
2. *Any family included in a linearly independent family is linearly independent.*
3. *$\{v\}$ is linearly independent if and only if $v \neq 0$.*
4. *Any set containing the null vector is linearly dependent.*

2.5 Basis

A basis of a vector space is linearly independent generating familly.
If $B = (x_i)_{i\in I}$, $I \subset \mathbb{N}$ is a basis of E, then any $x \in E$ is uniquely written as a linear combination of elements of B.

$$x = \sum_{i\in I} \alpha_i x_i$$

The scalars $(\alpha_i)_{i\in I}$, are called the coordinates of x in the basis B.

3 Finite dimensional vector spaces

Definition 3.0.1 *If a vector space is spanned by a finite number of vectors, it is said to be* ***finite-dimensional****.*
Otherwise it is ***infinite-dimensional****. The number of vectors in a basis for a finite-dimensional vector space E is called the* ***dimension*** *of E and denoted dim$E$.*
By convention, we say that $\{0_E\}$ is a finite-dimensional space.

Definition 3.0.2 *A family $\{v_1, \cdots, v_n\}$ of vectors of E is said to be a basis of E if and only if, we have :*

1. *$\{v_1, \cdots, v_n\}$ is a linearly independent family of E and*
2. *$\{v_1, \cdots, v_n\}$ is a generating family of E.*

Example 3.0.1

1. *The set $(1, i)$ is a basis of the $\mathbb{R}-$vector space $\mathbb{C}$.*
   Indeed, if $a, b \in \mathbb{R}$ are such that $a.1 + b.i = 0$ then $a + ib = 0 + i0$ and therefore $a = b = 0$. The set is therefore linearly independent.
   For any complex number, there are $a, b \in \mathbb{R}$ such that $z = a+ib$, then $(1, i)$ is a generating set of $\mathbb{C}$, it is therefore a basis of $\mathbb{C}$.
2. *In $\mathbb{R}^3$, the set $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ forms a basis of $\mathbb{R}^3$, called* ***canonical basis*** *of $\mathbb{R}^3$.*
   The set $\{v_1 = (1, 0, 1), v_2 = (1, -1, 1) v_3 = (0, 1, 1)\}$ is a basis of $\mathbb{R}^3$. Indeed :

a/ *The family is linearly independent.*
Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$. Then

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

which leads to $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

b/ *The set is generating of* $\mathbb{R}^3$. *Let* $(x, y, z) \in \mathbb{R}^3$. *We are looking for* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *such that* $(x, y, z) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. *We then obtain the system*

$$\begin{aligned} \alpha_1 + \alpha_2 \quad &= x \\ \alpha_2 + \alpha_3 \quad &= y \\ \alpha_1 + \alpha_2 + \alpha_3 \quad &= z \end{aligned}$$

and we find $\alpha_1 = 2x + y - z, \alpha_2 = x - y + z$ *and* $\alpha_3 = -x + z$.
So span $\{v_1 = (1, 0, 1), v_2 = (1, -1, 1), v_3 = (0, 1, 1)\} = \mathbb{R}^3$. *Then* $\{v_1, v_2, v_3\}$ *is a basis of* $\mathbb{R}^3$.

More generally, we have :

Proposition 3.0.1 Canonical base of $\mathbb{K}^n$
Consider the vector space $E = \mathbb{K}^n$ *over the field* $\mathbb{K}$.
The standard basis vectors of E *are a specific set of basis vectors that are commonly used in linear algebra. They are the unit vectors in each dimension of the vector space :*
$(e_1, e_2, \cdots, e_n)$ *of* $\mathbb{K}^n$ *called* **canonical** *and given by :*

$$e_1 = (1, 0, 0, \cdots, 0), e_2 = (0, 1, 0, 0, \cdots, 0), \cdots, e_n = (0, 0, 0, \cdots 0, 1).$$

Proposition 3.0.2 Canonical base of $\mathbb{K}_n[X]$
Let $n \in \mathbb{N}$. *Consider the vector space* $E = \mathbb{K}_n[X]$ *of polynomials of degree* $\leq n$ *with coefficients in* $\mathbb{K}$. *There is a specific basis of* $\mathbb{K}_n[X]$ *called* ***canonical****, given by* $\{1, X, X^2, \cdots, X^n\}$.

Theoreme 3.0.1 Theorem of the extracted basis *From any finite generating family of* E, *we can extract a basis of* E. *In particular, a finite-dimensional space admits a basis.*

Theoreme 3.0.2 Incomplete basis theorem *If* E *is finite-dimensional, then any linearly independent family of* E *can be completed into a basis of* E. *To complete it, simply consider certain vectors of a generating family of* E.

Theoreme 3.0.3 Dimension *If* E *is finite-dimensional, then all bases of* E *have the same number of vectors (dimension of* E*).*

Corollary 3.0.1 *If* E *is a finite-dimensional vector space (*$dimE = n$*) and if* $B = (v_1, v_2, \cdots, v_n)$ *is a family of* n *vectors of* E, *then the following conditions are equivalent :*

1. B *is linearly independent.*
2. B *is a generating set of* E.
3. B *is a basis of* E.

Remark 3.0.1
1. *In particular, in a* n*-dimensional space, a linearly independent set always has at most* n *elements, and a generating family always has at least* n *elements.*
2. *If* E *and* F *are finite-dimensional, then* $dim(E \times F) = dim(E) + dim(F)$. *In particular,* $dim(\mathbb{K}^n) = n$.
3. $dim(\mathbb{K}_n[X]) = n + 1$.

Definition 3.0.3 *If* $(v_1, v_2, \cdots, v_n)$ *is a finite set of* E, *we call* ***rank*** *of* $(v_1, v_2, \cdots, v_n)$ *the dimension of* $F = Vect(v_1, v_2, \cdots, v_n)$.

Let $G = \{v_1 = (2, 1), v_2 = (4, 2), v_3 = (-3, 4)\}$ *be a subset of* $\mathbb{R}^2$. *Let's determine the rank of* G.
The set G *is linearly dependent* $(v_2 = 2v_1)$, *so span* $(v_1, v_2, v_3) = span(v_2, v_3)$, *so* $rank(G) = 2$.

3.1 Subspaces and dimension

If E is a finite-dimensional vector space and if F is a subspace of E, then we have $dim(F) \leq dim(E)$ and Furthermore :

$$dim(F) = dim(E) \Leftrightarrow F = E.$$

Grassmann formula *: Let E be a finite-dimensional vector space and let F, G be two subspaces of E. Then*

$$dim(F + G) = dim(F) + dim(G) - dim(F \cap G).$$

In particular, F and G are in direct sum if and only if

$$dim(F + G) = dim(F) + dim(G).$$