FINAL EXAM ANSWERS

Exercise 1 *Are the following statements true or false? Justify your answers.*

1. ***(T)***. *Suppose that* $\left(\forall n \in \mathbb{N}^*, \exists p \in \mathbb{N}^* : n = p^2\right) \wedge \left(\exists q \in \mathbb{N}^* / \quad 2n = q^2\right)$.
   Let $n \in \mathbb{N}^*, \quad n = p^2$ *and* $\quad q^2 = 2n$, *then* $2p^2 = q^2 (p, q \in \mathbb{N}^*)$, *i.e.* $\sqrt{2} = \frac{p}{q} \in \mathbb{Q}$ *contradiction (* $\sqrt{2}$ *is an irrational number).*
2. ***(F)***. *The relation* $\mathcal{R}$ *defined by :* $x\mathcal{R}y \Leftrightarrow \cos^2 x + \sin^2 y = 1$ *is not anti-symmetrical.*
   We have $0\mathcal{R}2\pi$ *and* $2\pi\mathcal{R}0$ *but* $0 \neq 2\pi$.
3. ***(T)***. $(x\mathcal{R}y \Leftrightarrow x - y$ *is a multiple of* $5)$, $\overline{2024} = \overline{2020 + 4} = \overline{0 + 4} = \overline{4}$ *and* $3 \notin \overline{4}$ *so* $\overline{3} \cap \overline{4} = \emptyset$.
4.

a/ ***(T)***. *Let* $x \in B \subset A \cup B \subset A \cup C$, *then* $x \in A \vee x \in C$.
If $x \in A$ *(and* $x \in B$*) then* $x \in A \cap B \subset A \cap C$ *then* $x \in C$.
So $(\forall x \in B, \quad x \in C)$ *which implies that* $B \subset C$.

b/ ***(F)***. *For example :* $A = \{0\}$, $B = \{0, 1\}$ *and* $C = \{1\}$.

Exercise 2 *Consider the map* $f : \mathbb{R} \to \mathbb{R}$ *defined by :* $f(x) = \sqrt{1 + x^2}$.

1.

$$\begin{aligned} f^{-1}(\{0\}) &= \{x \in \mathbb{R} / \quad f(x) \in \{0\}\} \\ &= \{x \in \mathbb{R} / \quad f(x) = 0\} \\ &= \left\{x \in \mathbb{R} / \quad x^2 = -1\right\} = \emptyset. \end{aligned}$$

$$\begin{aligned} f^{-1}(\{2\}) &= \{x \in \mathbb{R} / \quad f(x) = 2\} \\ &= \left\{x \in \mathbb{R} / \quad \sqrt{1 + x^2} = 2\right\} \\ &= \left\{x \in \mathbb{R} / \quad x^2 = 3\right\} \\ &= \left\{-\sqrt{3}, \sqrt{3}\right\}. \end{aligned}$$

$f\left(\left\{-\sqrt{3}, 0, \sqrt{3}\right\}\right) = \left\{f(x) / \quad x \in \left\{-\sqrt{3}, 0, \sqrt{3}\right\}\right\}$.
We have : $f(-\sqrt{3}) = f(\sqrt{3}) = 2$ *and* $f(0) = 1$. *So*
$f\left(\left\{-\sqrt{3}, 0, \sqrt{3}\right\}\right) = \{1, 2\}$.

2. *We have :* $f(-\sqrt{3}) = f(\sqrt{3}) = 2$ *but* $-\sqrt{3} \neq \sqrt{3}$ *then the map* f *is not injective.*
3. $f^{-1}(\{0\}) = \emptyset$, *then* $\nexists x \in \mathbb{R} / \quad f(x) = 0$, *then the map* f *is not surjective.*
4. *Let* $\mathcal{R}$ *the binary relation defined by :* $\forall x \in \mathbb{R}, \quad x\mathcal{R}y \Leftrightarrow f(x) = f(y)$.

a/ $\mathcal{R}$ *is an equivalence relation :*

- $\mathcal{R}$ ***reflexive*** : $\forall x \in \mathbb{R}, \quad f(x) = f(x)$.
- $\mathcal{R}$ ***symmetrical*** : $\forall x, y \in \mathbb{R}, \quad f(x) = f(y) \Rightarrow f(y) = f(x)$.
- $\mathcal{R}$ ***transitive*** : $\forall x, y, z \in \mathbb{R}, (f(x) = f(y)) \wedge (f(y) = f(z)) \Rightarrow f(x) = f(z)$.

b/

$$\begin{aligned} \overline{\sqrt{2}} &= \left\{x \in \mathbb{R} / \quad x\mathcal{R}\sqrt{2}\right\} \\ &= \left\{x \in \mathbb{R} / \quad f(x) = f(\sqrt{2})\right\} \\ &= \left\{x \in \mathbb{R} / \quad 1 + x^2 = 3\right\} \\ &= \left\{-\sqrt{2}, \sqrt{2}\right\}. \end{aligned}$$

5. *Let the map* $g:[0,+\infty[\ \to [1,+\infty[$ *defined by* $g(x)=\sqrt{1+x^2}$.

a/ *g is a bijection :* $\forall y\in[1,+\infty[,\exists!?x\in[0,+\infty[/\quad y=\sqrt{1+x^2}$.
We have $y=\sqrt{1+x^2}\Rightarrow x^2=y^2-1\Rightarrow x=\pm\sqrt{y^2-1}$, *but* $-\sqrt{y^2-1}\notin[0,+\infty[$, *then :*
$\forall y\in[1,+\infty[,\exists!$ *(a unique)* $x=\sqrt{y^2-1}\in[0,+\infty[/\quad y=g(x)$. *Then g is a bijection.*

b/ $$\begin{array}{llll} g^{-1}: & [1,+\infty[& \to & [0,+\infty\} \\ & x & \longmapsto & g^{-1}(x)=\sqrt{x^2-1} \end{array}$$

Exercise 3 *On the set* $G=\mathbb{R}\setminus\{-1\}$, *consider the following composition law :*

$$x*y=xy+x+y$$

1. $x*y\neq -1,\quad for\quad all\quad x,y\in G:$
Suppose that $x*y=-1$, *then* $xy+x+y=-1$ *i.e.* $x(y+1)=-(y+1)\Rightarrow x=\frac{-(y+1)}{y+1},\quad y\neq -1$ *then* $x=-1$ *contradiction with* $x\in\mathbb{R}\setminus\{-1\}$.
2. *G is an abelian group :*

i/ *The law* $(*)$ *is commutative :* $\forall x,y\in G,\quad x*y=xy+x+y=yx+y+x=y*x$.

ii/ *The law* $(*)$ *is associative :* $\forall x,y,z\in G,\quad (x*y)*z=(xy+x+y)*z=(xy+x+y)z+(xy+x+y)+z=xyz+xz+yz+xy+x+y+z=x(yz+y+z)+x+(yz+y+z)=x(y*z)+x+(y*z)=x*(y*z)$.

iii/ $\exists?e\in G/\quad \forall x\in G, x*e=e*x=x$. $(*)$ *is commutative then we simply find* $e\in G$ *such that* $x*e=x$ *so* $xe+x+e=x\Leftrightarrow e(x+1)=0,\quad x\neq -1$ *then* $e=0$.

iv/ $\forall x\in G,\exists?x'\in G/\quad x*x'=x'*x=e=0$. $(*)$ *is commutative then,* $\forall x\in G$, *we simply find* $x'\in G$ *such that* $x*x'=0$ *so* $xx'+x+x'=0\Leftrightarrow x'(x+1)+x=0\Rightarrow x'=\frac{-x}{x+1}$ *with* $x\neq -1$

3. *Let* $H=\{x\in\mathbb{R},\quad x>-1\}$. $(H,*)$ *is a subgroup of* $(G,*)$:

i/ $(e=0)>-1$ *then* $0\in H$.

ii/ $\forall x,y\in H, x*y\in?H$.
We have $x,y\in H\Leftrightarrow x>-1$ *and* $y>-1$ *then* $x*y=xy+x+y=x(y+1)+(y+1)-1=(x+1)(y+1)-1>-1$, *since* $(x+1)(y+1)>0$. *Then* $x*y\in H$.

iii/ $\forall x\in H(x>-1),\quad x^{-1}\in?H$.
We have $x^{-1}=\frac{-x}{x+1}>-1\Leftrightarrow\frac{-x}{x+1}+1>0$. *Then* $\frac{-x}{x+1}+1=\frac{-x}{x+1}+\frac{x+1}{x+1}=\frac{1}{x+1}>0$ *since* $x>-1$.

4. *Let* $f:(G,*)\to(R\setminus\{0\},\cdot)$ *be the application defined by* $f(x)=x+1$.

a/ *We have* $\forall x,y\in G,\quad f(x*y)=f(xy+x+y)=(xy+x+y)+1=(x+1)\cdot(y+1)=f(x)\cdot f(y)$. *Then f is an homomorphism of groups.*

5. *We show by induction that*

$$\forall n\geq 2,\quad \underbrace{x*x*\cdots*x}_{n\,times}=(x+1)^n-1$$

a/ ***Base case*** *:For* $n=2$, *we have :* $(x*x)=x^2+x+x=(x+1)^2-1$.

b/ ***Inductive step*** *: Assume that* $\underbrace{x*x*\cdots*x}_{n\,times}=(x+1)^n-1$ *and show that* $\underbrace{x*x*\cdots*x}_{(n+1)\,times}=(x+1)^{n+1}-1$.

We have :

$$\begin{aligned}\underbrace{x*x*\cdots*x}_{(n+1)\,times} &= \underbrace{x*x*\cdots*x}_{n\,times}*x \\ &= ((x+1)^n-1)*x \\ &= ((x+1)^n-1)x+((x+1)^n-1)+x \\ &= (x+1)^n\cdot x-x+(x+1)^n-1+x \\ &= (x+1)^n x+(x+1)^n-1=(x+1)^n\cdot(x+1)-1 \\ &= (x+1)^{n+1}-1.\end{aligned}$$