Course : Algebra 3 **Year : 2023/2024**
Chapter 1 : Determinants of matrices **Department of Computer Science**

Solutions

Solution 0.1 .

1)

1. *A is a symmetric matrix if* $a = 5$ *and* $b = 6$.

2. *A is an upper triangular matrix if* $a = b = 0$.

2) C is a diagonal matrix if a, e *and* i *are not equal to* 0 *such that* $b = c = f = d = g = h = 0$.

3) $B^T = \begin{pmatrix} 3 & 0 & 4 \\ 4 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}$.

4)$|B| = 3\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 31 \neq 0$ *yes, it is possible to determine* B^{-1}

Solution 0.2 .

$|A| = 5\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = -21 \neq 0$. *We deduce that* A^{-1} *exists.*

Solution 0.3 .

$$\begin{aligned} |A| &= \alpha\begin{vmatrix} 2 & \alpha \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & \alpha \\ (\alpha-2) & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ (\alpha-2) & 1 \end{vmatrix} \\ &= -4(\alpha - 2) \end{aligned}$$

This leads to $|A| = 0 \Longrightarrow \alpha = 2$

Solution 0.4 .

1)

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 7 \\ 0 & 1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

.

$|A| = 46.$

$x = \dfrac{\begin{vmatrix} 3 & 1 & 3 \\ 0 & 5 & 7 \\ 2 & 1 & 8 \end{vmatrix}}{46} = \dfrac{83}{46}, y = \dfrac{\begin{vmatrix} 2 & 3 & 3 \\ 4 & 0 & 7 \\ 0 & 2 & 8 \end{vmatrix}}{46} = \dfrac{-100}{46} = \dfrac{-50}{23}, z = \dfrac{\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 0 & 1 & 2 \end{vmatrix}}{46} = \dfrac{24}{46}.$

2)

$$\begin{pmatrix} 3 & 2 & 9 \\ 8 & 0 & 1 \\ 7 & 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

.

$|A| = 295.$

$x = \dfrac{\begin{vmatrix} 4 & 2 & 9 \\ 2 & 0 & 1 \\ 1 & 5 & 4 \end{vmatrix}}{295} = \dfrac{56}{295}, y = \dfrac{\begin{vmatrix} 3 & 4 & 9 \\ 8 & 2 & 1 \\ 7 & 1 & 4 \end{vmatrix}}{295} = \dfrac{-133}{295}, z = \dfrac{\begin{vmatrix} 3 & 2 & 4 \\ 8 & 0 & 2 \\ 7 & 5 & 1 \end{vmatrix}}{295} = \dfrac{142}{295}.$

Solution 0.5 .

1)

$$\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

$|A| = 17.$

$A^{-1} = \dfrac{1}{17} \begin{pmatrix} 4 & -1 \\ -3 & 5 \end{pmatrix}.$

$X = A^{-1}B = \dfrac{1}{17} \begin{pmatrix} 4 & -1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{14}{17} \\ \frac{15}{17} \end{pmatrix}.$

2)

$$\begin{pmatrix} 1 & 2 & 4 \\ 7 & 5 & 3 \\ 9 & 7 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

$|A| = 40.$

$A^{-1} = \dfrac{1}{40} \begin{pmatrix} -16 & 26 & -14 \\ 20 & -35 & 25 \\ 4 & 11 & -9 \end{pmatrix}.$

$X = A^{-1}B = \begin{pmatrix} \frac{6}{40} \\ \frac{15}{40} \\ \frac{1}{40} \end{pmatrix}.$

Solution 0.6 .

1) We have $A^2 = A$ *and* $B = A - I$. *This leads to*

$$\begin{aligned} B^2 &= (A-I)^2 \\ &= A^2 - 2A + I \\ &= A - 2A + I \\ &= -A + I \\ &= -(A - I) \\ &= -B \end{aligned}$$

2) We have $A^2 = I$ *and* $B = 3(A+I)$*. This leads to*

$$\begin{aligned} B^2 &= (3(A+I))^2 \\ &= 9(A^2+2A+I) \\ &= 9(2I+2A) \\ &= 6(3(I+A)) \\ &= 6B \end{aligned}$$

Solution 0.7 .

1)Take that

$$\begin{aligned} (A^{-1})^T A^T &= (AA^{-1})^T \\ &= (I)^T \\ &= I \end{aligned}$$

and that

$$\begin{aligned} A^T (A^{-1})^T &= (A^{-1}A)^T \\ &= (I)^T \\ &= I \end{aligned}$$

Thus, we find

$$(A^{-1})^T A^T = I, \text{ and } A^T (A^{-1})^T = I.$$

In the sequel, we deduce

$$(A^T)^{-1} = (A^{-1})^T.$$

2).

a) From the property of the determinant of the product of two matrices, we have

$$|AB| = |A||B|$$

. *We know that* $|A| \neq 0$ *and* $|B| \neq 0$*. This leads to*

$$|AB| \neq 0$$

.

b)We have $|AB| \neq 0$*, then* AB *is invertible. Here, we take that*

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I\end{aligned}$$

and that

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A(I)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Thus, we obtain

$$(B^{-1}A^{-1})(AB) = I, \text{ and } (AB)(B^{-1}A^{-1}) = I$$

Then, we deduce that

$$(AB)^{-1} = B^{-1}A^{-1}.$$