Course : Algebra 3 **Year : 2023/2024**
Chapter 2 : Linear systems **Department of Computer Science**

Solutions

Solution 0.1 .

1)

$$A = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 3 & 7 \\ 8 & -4 & -12 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -1 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

the rank of A is 2.

2)

$$B = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 9 & 0 & 5 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -5 & -7 & -9 \\ 0 & -27 & -31 & -44 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -5 & -7 & -9 \\ 0 & 0 & 34/5 & 23/5 \end{pmatrix},$$

the rank of B is 3.

3)

$$C = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & 0 \\ 5 & 5 & 15/2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & -3 & -3/2 \\ 0 & 0 & 0 \end{pmatrix},$$

the rank of C is 2.

4)

$$D = \begin{pmatrix} 3 & 4 & -5 \\ 1 & 4/3 & -5/3 \\ -4 & -16/3 & 20/3 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the rank of D is 1.

Solution 0.2 .

1) The augmented matrix is defined by

$$\begin{pmatrix} 1 & 3 & 5 & 6 & 1 \\ 4 & 1 & -2 & 3 & -2 \\ 1 & -3 & -7 & 8 & 3 \end{pmatrix}.$$

By using the Gauss elimination method, we get

$$\begin{pmatrix} 1 & 3 & 5 & 6 & 1 \\ 0 & -11 & -22 & -21 & -6 \\ 0 & -6 & -12 & 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 5 & 6 & 1 \\ 0 & -11 & -22 & -21 & -6 \\ 0 & 0 & 0 & 148/11 & 58/11 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} x_1 + 3x_2 + 5x_3 + 6x_4 &= 1 \\ -11x_2 - 22x_3 - 21x_4 &= -6 \\ \frac{148}{11}x_4 &= \frac{58}{11} \end{aligned}$$

This leads to $x_1 = -\frac{1}{2}x_2 - \frac{125}{148},\ x_3 = -\frac{165}{1628} - \frac{814}{1628}x_2 = -\frac{15}{148} - \frac{1}{2}x_2,\ x_4 = \frac{58}{148} = \frac{29}{74}$.

2) The augmented matrix is defined by

$$\begin{pmatrix} 2 & 3 & 4 & 3 \\ 2 & 5 & 4 & 5 \\ -2 & 1 & -7 & 1 \end{pmatrix}.$$

By using the Gauss elimination method, we get

$$\begin{pmatrix} 2 & 3 & 4 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 4 & -3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 3 & 4 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 3 \\ 2x_2 &= 2 \\ -3x_3 &= 0 \end{aligned}$$

This leads to $x_1 = 0,\ x_2 = 1,\ x_3 = 0.$

3) The augmented matrix is defined by

$$\begin{pmatrix} 4 & 2 & 4 & 7 & 9 & 6 & 1 \\ 5 & 6 & 1 & 0 & 3 & 7 & 3 \\ 4 & 5 & 2 & 3 & 1 & 0 & 1 \end{pmatrix}.$$

By using the Gauss elimination method, we get

$$\begin{pmatrix} 4 & 2 & 4 & 7 & 9 & 6 & 1 \\ 0 & 7/2 & -4 & -35/4 & -33/4 & -1/2 & 7/4 \\ 0 & 3 & -2 & -4 & -8 & -6 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 2 & 4 & 7 & 9 & 6 & 1 \\ 0 & 7/2 & -4 & -35/4 & -33/4 & -1/2 & 7/4 \\ 0 & 0 & 10/7 & 7/2 & -13/14 & -39/7 & -3/2 \end{pmatrix}.$$

This leads to

$$\begin{aligned} x_1 &= \frac{33}{20} + \frac{17}{20}x_4 - \frac{89}{20}x_5 - \frac{77}{10}x_6 \\ x_2 &= -\frac{7}{10} - \frac{3}{10}x_4 + \frac{31}{10}x_5 + \frac{23}{5}x_6 \\ x_3 &= -\frac{21}{20} - \frac{49}{20}x_4 + \frac{13}{20}x_5 + \frac{39}{10}x_6 \end{aligned}$$

Solution 0.3 . *By applying the Gauss elimination method, we obtain*

$$A = \begin{pmatrix} 5 & 6 & 1 \\ 10 & 2 & 4 \\ -5 & 3\beta - 2 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 5 & 6 & 1 \\ 0 & -10 & 2 \\ 0 & 3\beta + 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 5 & 6 & 1 \\ 0 & -10 & 2 \\ 0 & 0 & (3\beta + 24)/5 \end{pmatrix}.$$

The rank of A is equal to 3 when $\frac{3\beta + 24}{5} \neq 0$. *Then,* $\beta \neq -8$.

Solution 0.4 . *By the Gauss elimination method, we have*

$$\begin{pmatrix} 5 & 5 & 3 & 2 \\ 7 & 2 & \alpha/2 & 14/5 \\ 2 & 1 & 3/5 & 2/5 \end{pmatrix},$$

$$\begin{pmatrix} 5 & 5 & 3 & 2 \\ 0 & -5 & (5\alpha-42)/10 & 0 \\ 0 & -1 & -3/5 & -2/5 \end{pmatrix},$$

$$\begin{pmatrix} 5 & 5 & 3 & 2 \\ 0 & -5 & (5\alpha-42)/10 & 0 \\ 0 & 0 & 6/25-\alpha/10 & -2/5 \end{pmatrix}.$$

The system has a unique solution if $6/25 - \alpha/10 \neq 0$. *Then,* $\alpha \neq 12/5$.
The system does not have solutions if $6/25 - \alpha/10 = 0$. *Then,* $\alpha = 12/5$.

Solution 0.5 .

By the Gauss elimination method, we find

$$\begin{pmatrix} 3 & 1 & 4 & 5 & 7 \\ 3 & 0 & 2 & 0 & 7 \\ 6 & 5 & 0 & 1 & 0 \\ 9 & 6 & 2 & -\alpha+5 & 5 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 4 & 5 & 7 \\ 0 & -1 & -2 & -5 & 0 \\ 0 & 3 & -8 & -9 & -14 \\ 0 & 3 & -10 & -\alpha-10 & -16 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 4 & 5 & 7 \\ 0 & -1 & -2 & -5 & 0 \\ 0 & 0 & -14 & -24 & -14 \\ 0 & 0 & -16 & -\alpha-25 & -16 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 4 & 5 & 7 \\ 0 & -1 & -2 & -5 & 0 \\ 0 & 0 & -14 & -24 & -14 \\ 0 & 0 & 0 & -\alpha+17/7 & 0 \end{pmatrix},$$

The given system has a unique solution if $-\alpha + 17/7 \neq 0$. *Then,* $\alpha \neq 17/7$.
The given system has infinite solutions if $-\alpha + 17/7 = 0$. *Then,* $\alpha = 17/7$.