Course : Algebra 3 **Year : 2023/2024**
Chapter 3 : Endomorphisms **Department of Computer Science**

Solutions

Solution 0.1 .
1) The matrix D is similar to A, then

$$D = P^{-1}AP,$$

such that P is a square matrix and is nonsingular. This leads to

$$\begin{aligned}\det D &= \det(P^{-1}AP)\\ &= \det(P^{-1})(\det A)(\det P)\\ &= \frac{1}{\det P}(\det A)(\det P)\\ &= \det A\end{aligned}$$

2) We have that X is an eigenvector of A

$$AX = \lambda X.$$

Then, we find

$$PAP^{-1}X = \lambda X.$$

This yields

$$D(P^{-1})X = \lambda(P^{-1}X).$$

If we take that

$$Z = P^{-1}X,$$

we get

$$DZ = \lambda Z.$$

Finally, we deduce that $Z = P^{-1}X$ is an eigenvector of D.

Solution 0.2 . *We have that A and D are similar, then*

$$D = P^{-1}AP,$$

which leads to

$$\begin{aligned}\det(D-\lambda I) &= \det(P^{-1}(A-\lambda I)P)\\ &= \det(P^{-1})\det(A-\lambda I)\det P\\ &= \frac{1}{\det P}\det(A-\lambda I)\det P\\ &= \det(A-\lambda I)\end{aligned}$$

Solution 0.3 .

$$\begin{aligned}\det(A-\lambda I) &= \det(A-\lambda I)^T\\ &= \det(A^T-\lambda I^T)\\ &= \det(A^T-\lambda I)\end{aligned}$$

Solution 0.4 .

1) The characteristic polynomial of A is given by

$$\begin{aligned} p(\lambda) &= \det(A-\lambda I) \\ &= \begin{vmatrix} 4-\lambda & 1 \\ 9 & 4-\lambda \end{vmatrix} \\ &= (4-\lambda)^2 - 9 \\ &= \lambda^2 - 8\lambda + 7 \end{aligned}$$

2) We take

$$p(\lambda) = \lambda^2 - 8\lambda + 7 = (\lambda-1)(\lambda-7) = 0,$$

then, the eigenvalues of A are $\lambda_1 = 1$ *and* $\lambda_2 = 7$.

The eigenvectors of A are given by

If $\lambda_1 = 1$*, then we get*

$$(A-\lambda_1 I)X_1 = \begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This yields

$$\begin{aligned} 3x_1 + x_2 &= 0 \\ 9x_1 + 3x_2 &= 0 \end{aligned}$$

Here, we have $x_2 = -3x_1$*. Then, the eigenvector corresponding to* λ_1 *is given by*

$$X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

If $\lambda_2 = 7$*, then we get*

$$(A-\lambda_2 I)X_2 = \begin{pmatrix} -3 & 1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This yields

$$\begin{aligned} -3x_1 + x_2 &= 0 \\ 9x_1 - 3x_2 &= 0 \end{aligned}$$

Here, we have $x_2 = 3x_1$*. Then, the eigenvector corresponding to* λ_2 *is given by*

$$X_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

3) Yes, the matrix A is diagonalizable.

4) The matrix P is

$$P = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix},$$

and

$$P^{-1} = \frac{1}{6} \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix},$$

such that

$$D=\begin{pmatrix}1&0\\0&7\end{pmatrix}.$$

Finally, we obtain

$$P^{-1}AP=\begin{pmatrix}\frac{1}{2}&-\frac{1}{6}\\\frac{1}{2}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}4&1\\9&4\end{pmatrix}\begin{pmatrix}1&1\\-3&3\end{pmatrix}=\begin{pmatrix}1&0\\0&7\end{pmatrix}.$$

Solution 0.5 .

$$\begin{aligned}\det(A-\lambda I)&=\begin{vmatrix}1-\lambda&2&-3\\1&1-\lambda&2\\1&0&3-\lambda\end{vmatrix}\\&=(1-\lambda)(\lambda-2)^2\end{aligned}$$

Then, the eigenvalues of A are $\lambda_1=1$ and $\lambda_2=\lambda_3=2$. Now, we need to find the eigenvectors of A.

If $\lambda_1=1$, we get

$$(A-\lambda_1 I)X_1=\begin{pmatrix}0&2&-3\\1&0&2\\1&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Here, we have $x_1=-2x_3, x_2=\frac{3}{2}x_3$. Then

$$X_1=\begin{pmatrix}-2\\\frac{3}{2}\\1\end{pmatrix}.$$

If $\lambda_2=\lambda_3=2$, we get

$$(A-\lambda_2 I)X_2=\begin{pmatrix}-1&2&-3\\1&-1&2\\1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Here, we have $x_1=-x_3$ and $x_2=x_3$. Then,

$$X_2=\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

Finally, we deduce that A is not diagonalizable.

Solution 0.6 .

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} -1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 4 \\ -9 & 4 & -3-\lambda \end{vmatrix} \\ &= (1+\lambda)(4-\lambda)(4+\lambda)\end{aligned}$$

Then, the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 4$ and $\lambda_3 = -4$. Now, we need to find the eigenvectors of A.

If $\lambda_1 = -1$, we get

$$(A-\lambda_1 I)X_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 4 & 4 \\ -9 & 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_2 = -x_3$ and $x_1 = \frac{-2}{3}x_3$. Then

$$X_1 = \begin{pmatrix} \frac{-2}{3} \\ -1 \\ 1 \end{pmatrix}.$$

If $\lambda_2 = 4$, we get

$$(A-\lambda_2 I)X_2 = \begin{pmatrix} -5 & 1 & 1 \\ 0 & -1 & 4 \\ -9 & 4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_2 = 4x_3$ and $x_1 = x_3$. Then,

$$X_2 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}.$$

If $\lambda_3 = -4$, we get

$$(A-\lambda_3 I)X_3 = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 7 & 4 \\ -9 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_2 = -\frac{4}{7}x_3$ and $x_1 = -\frac{1}{7}x_3$. Then,

$$X_3 = \begin{pmatrix} -\frac{1}{7} \\ -\frac{4}{7} \\ 1 \end{pmatrix}.$$

Then, the matrix P is

$$P = \begin{pmatrix} -\frac{2}{3} & 1 & -\frac{1}{7} \\ -1 & 4 & -\frac{4}{7} \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$P^{-1} = -\frac{21}{40}\begin{pmatrix} \frac{32}{7} & -\frac{8}{7} & 0 \\ \frac{3}{7} & -\frac{11}{21} & -\frac{5}{21} \\ -5 & \frac{5}{3} & -\frac{5}{3} \end{pmatrix}.$$

Finally, we obtain

$$P^{-1}AP = \begin{pmatrix} -\frac{672}{280} & \frac{168}{280} & 0 \\ -\frac{63}{280} & \frac{11}{40} & \frac{5}{40} \\ \frac{105}{40} & -\frac{105}{120} & \frac{105}{120} \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 4 \\ -9 & 4 & -3 \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & 1 & -\frac{1}{7} \\ -1 & 4 & -\frac{4}{7} \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Solution 0.7 .

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 3 & 1 \\ 0 & 4-\lambda & 2 \\ 26 & 24 & 6-\lambda \end{vmatrix} \\ &= (\lambda+2)(\lambda+1)(14-\lambda) \end{aligned}$$

Then, the eigenvalues of A are $\lambda_1 = -2, \lambda_2 = -1$ and $\lambda_3 = 14$. Now, we need to find the eigenvectors of A.

If $\lambda_1 = -2$, we get

$$(A-\lambda_1 I)X_1 = \begin{pmatrix} 3 & 3 & 1 \\ 0 & 6 & 2 \\ 26 & 24 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_3 = -3x_2$ and $x_1 = 0$. Then

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

If $\lambda_2 = -1$, we get

$$(A-\lambda_2 I)X_2 = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 5 & 2 \\ 26 & 24 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_3 = -\frac{5}{2}x_2$ and $x_1 = -\frac{1}{4}x_2$. Then,

$$X_2 = \begin{pmatrix} -\frac{1}{4} \\ 1 \\ -\frac{5}{2} \end{pmatrix}.$$

If $\lambda_3 = 14$, we get

$$(A-\lambda_3 I)X_3 = \begin{pmatrix} -13 & 3 & 1 \\ 0 & -10 & 2 \\ 26 & 24 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we have $x_3 = 5x_2$ *and* $x_1 = \frac{8}{13}x_2$. *Then,*

$$X_3 = \begin{pmatrix} \frac{8}{13} \\ 1 \\ 5 \end{pmatrix}.$$

Then, the matrix P *is*

$$P = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{8}{13} \\ 1 & 1 & 1 \\ -3 & -\frac{5}{2} & 5 \end{pmatrix},$$

and

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 14 \end{pmatrix}.$$

Here, we have

$$Y^{'} = DY,$$

Then

$$Y = \begin{pmatrix} C_1 \exp(-2t) \\ C_2 \exp(-t) \\ C_3 \exp(14t) \end{pmatrix},$$

Finally, the solution is

$$X = PY = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{8}{13} \\ 1 & 1 & 1 \\ -3 & -\frac{5}{2} & 5 \end{pmatrix} \begin{pmatrix} C_1 \exp(-2t) \\ C_2 \exp(-t) \\ C_3 \exp(14t) \end{pmatrix}.$$

Then

$$X = \begin{pmatrix} -\frac{1}{4}C_2 \exp(-t) + \frac{8}{13}C_3 \exp(14t) \\ C_1 \exp(-2t) + C_2 \exp(-t) + C_3 \exp(14t) \\ -3C_1 \exp(-2t) - \frac{5}{2}C_2 \exp(-t) + 5C_3 \exp(14t) \end{pmatrix}.$$