Course : Algebra 3 **Year : 2023/2024**
Chapter 4 : Vector spaces **Department of Computer Science**

Solutions

Solution 0.1 .

1) For $x \in \mathbb{R}^n$, we find

$$< x, x >= \sum_{i=1}^{n} x_i^2 \succcurlyeq 0 \tag{1}$$

and when $< x, x >= 0$, we get

$$\sum_{i=1}^{n} x_i^2 = 0, \tag{2}$$

then, $x_1 = x_2 = \cdots = x_n = 0$. We deduce that $x = 0$.

2) For $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} < x, y > &= \sum_{i=1}^{n} x_i y_i \\ &= \sum_{i=1}^{n} y_i x_i \\ &= < y, x > \end{aligned}$$

3) For $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned} < x + y, z > &= \sum_{i=1}^{n} (x_i + y_i) z_i \\ &= \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i \\ &= < x, z > + < y, z > \end{aligned}$$

4) For $x, y \in \mathbb{R}^n$ and a real scalar α, we can get

$$\begin{aligned} < \alpha x, y > &= \sum_{i=1}^{n} \alpha x_i y_i \\ &= \alpha \sum_{i=1}^{n} x_i y_i \\ &= \alpha < x, y > \end{aligned}$$

Finally, we can say that $< x, y >$ is an inner product on $\mathbb{R}^n$.

Solution 0.2 .

1) For $f \in C[a,b]$, *we have*

$$< f, f >= \int_a^b w(t) f^2(t) dt \succcurlyeq 0, \tag{3}$$

because $w(t) \succ 0$ *and* $f^2(t) \succcurlyeq 0$ *and when*

$$\begin{aligned} < f, f >= 0 \quad &\Leftrightarrow \quad \int_a^b w(t) f^2(t) dt = 0 \\ &\Leftrightarrow \quad w(t) f^2(t) = 0 \\ &\Leftrightarrow \quad f(t) = 0 \end{aligned}$$

because $w(t) \succ 0$. *Then,* $< f, f >= 0 \Leftrightarrow f = 0$.

2) For $f, g \in C[a,b]$, *we have*

$$\begin{aligned} < f, g > \quad &= \quad \int_a^b w(t) f(t) g(t) dt \\ &= \quad \int_a^b w(t) g(t) f(t) dt \\ &= \quad < g, f > \end{aligned}$$

3) For $f, g, h \in C[a,b]$, *we have*

$$\begin{aligned} < f + g, h > \quad &= \quad \int_a^b w(t)(f+g)(t) h(t) dt \\ &= \quad \int_a^b w(t) f(t) h(t) dt + \int_a^b w(t) g(t) h(t) dt \\ &= \quad < f, h > + < g, h > \end{aligned}$$

4) For $f, g \in C[a,b]$ *and a scalar* α

$$\begin{aligned} < \alpha f, g > \quad &= \quad \int_a^b w(t)(\alpha f)(t) g(t) dt \\ &= \quad \alpha \int_a^b w(t) f(t) g(t) dt \\ &= \quad \alpha < f, g > \end{aligned}$$

Finally, we can say that $< f, g >$ *is an inner product on* $C[a,b]$.

Solution 0.3

$$\begin{aligned}
< f, g > &= \int_0^\pi f(t)g(t)dt \\
&= \int_0^\pi \cos(t)\sin(t)dt \\
&= \frac{1}{2}\int_0^\pi \sin(2t)dt \\
&= \frac{1}{2}[-\frac{\cos(2t)}{2}]_0^\pi \\
&= -\frac{1}{4}(\cos(2\pi) - \cos(0)) \\
&= -\frac{1}{4}(1-1) \\
&= 0
\end{aligned}$$

Then, $f(t)$ and $g(t)$ are orthogonal.

Solution 0.4 *We have $v_1 = 1, v_2 = x$ and $v_3 = x^2$. By employing Gram-Schmidt process, we get*

$$u_1 = v_1 = 1.$$

$$\begin{aligned}
u_2 &= v_2 - \frac{< v_2, u_1 >}{\| u_1 \|^2} u_1 \\
&= x - \frac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx} \\
&= x - \frac{1}{2}[\frac{x^2}{2}]_{-1}^1 \\
&= x
\end{aligned}$$

$$\begin{aligned}
u_3 &= v_3 - \sum_{i=1}^{2} \frac{< v_3, u_i >}{\| u_i \|^2} u_i \\
&= v_3 - \frac{< v_3, u_1 >}{\| u_1 \|^2} u_1 - \frac{< v_3, u_2 >}{\| u_2 \|^2} u_2 \\
&= x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 dx} - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2 dx} x \\
&= x^2 - \frac{1}{3}
\end{aligned}$$

Solution 0.5

$$\begin{aligned}\| \frac{1}{\sqrt{2}}(v_1 - v_2) \|^2 &= < \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(v_1 - v_2) > \\ &= \frac{1}{2}(< v_1, v_1 > - < v_2, v_1 > - < v_1, v_2 > + < v_2, v_2 >) \\ &= \frac{1}{2}(< v_1, v_1 > + < v_2, v_2 >) \\ &= 1\end{aligned}$$

$$\begin{aligned}\| \frac{1}{\sqrt{2}}(v_1 + v_2) \|^2 &= < \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(v_1 + v_2) > \\ &= \frac{1}{2}(< v_1, v_1 > + < v_2, v_1 > + < v_1, v_2 > + < v_2, v_2 >) \\ &= \frac{1}{2}(< v_1, v_1 > + < v_2, v_2 >) \\ &= 1\end{aligned}$$

$$< \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(v_1 + v_2) > \quad = \quad 0$$

Solution 0.6 .

1) For a scalar $\alpha \in Y$ and for $x, y \neq 0$

$$\begin{aligned}0 &\preccurlyeq \| x - \alpha y \|^2 \\ &= < x - \alpha y, x - \alpha y > \\ &= \| x \|^2 - \overline{\alpha} < x, y > - \alpha(< y, x > - \overline{\alpha} \| y \|^2).\end{aligned}$$

We take $\overline{\alpha} = \dfrac{< y, x >}{\| y \|^2}$. This leads to get

$$\begin{aligned}0 &\preccurlyeq \| x \|^2 - \frac{< y, x >< x, y >}{\| y \|^2} \\ &= \| x \|^2 - \frac{| < x, y > |^2}{\| y \|^2}.\end{aligned}$$

Then, we have

$$| < x, y > | \preccurlyeq \| x \| \| y \| \tag{4}$$

2) For $x, y \in V$

$$\begin{aligned}\| x + y \|^2 &= < x + y, x + y > \\ &= \| x \|^2 + < x, y > + < y, x > + \| y \|^2 \\ &\preccurlyeq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\ &= (\| x \| + \| y \|)^2.\end{aligned}$$

Then, we deduce that

$$\| x + y \| \preccurlyeq \| x \| + \| y \| \tag{5}$$