Algebra standard correction 1

Exercise 1 *1. Paris is in France or Madrid is in China. (T)*

2. *Open the door. Is not a proposition*
3. *The moon is a satelite of the earth. (T) Satellites are objects that revolve around other larger objects in space. Earth and the moon are called "natural" satellites.*
4. *x + 5 = 7. Is NOT a proposition ; as its truth value Unknown.*
5. $x + 5 > 9$ *for every real number* x*. (F)*

Exercise 2 *Determine whether each of the following implications is true or false.*

1. *If* 0.5 *is an integer, then* $1 + 0.5 = 3$*. (T)* $\quad (F \Rightarrow F)$
2. *If* $5 > 2$ *then cats can fly. (F)* $\quad (T \Rightarrow F)$.
3. *If* $3 \times 5 = 15$ *then* $1 + 2 = 3$*.(T)*
4. *For any real* $x \in \mathbb{R}$*, if* $x \leq 0$ *then* $(x - 1) < 0$*. (T)*

Exercise 3 *The statements* $P \Rightarrow (Q \vee R)$ *and* $(P \Rightarrow Q) \vee (P \Rightarrow R)$*. are logically equivalent, use the truth table.*
After simplification :

1. $P \wedge Q$.
2. $(P \wedge Q) \vee (Q \vee \overline{R})$.
3. *This is necessarly false, so it is equivalent to* $P \wedge \overline{P}$.

Exercise 4 *Consider the statement "for all integers a and b, if a + b is even, then a and b are even"*

1. *The contrapositive of the statement : For all integers a and b, if a or b is not even, then a + b is not even.*
2. *The converse of the statement : For all integers a and b, if a and b are even, then a + b is even.*
3. *The negation of the statement :There are numbers a and b such that a + b is even but a and b are not both even.*
4. *The original statement is false, For example,* $a = 3$ *and* $b = 5$*.* $a + b = 8$*, but neither a nor b are even.*
5. *The contrapositive of the original statement is false, since it is equivalent to the original statement.*
6. *The converse of the original statement is true. Let a and b be integers. Assume both are even. Then* $a = 2k$ *and* $b = 2j$ *for some integers k and j. But then* $a + b = 2k + 2j = 2(k + j)$ *which is even.*
7. *The negation of the original statement is true since the statement is false.*

Exercise 5 *1.* $(P \wedge Q) \wedge \overline{(P \vee Q)}$ *Contradiction.*

2. $\left((P \vee Q) \wedge \overline{P}\right) \Rightarrow Q$ *Tautology.*
3. $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \Rightarrow Q)$ *Contingency.*
4. $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ *Tautology.*

Exercise 6 *1.* $\forall x \in \mathbb{R}, \quad (x + 1)^2 = x^2 + 2x + 1$.

2. $\exists x \in \mathbb{R}, \quad x^2 + 3x + 2 = 0$.
3. $\exists! x \in \mathbb{R}/ \quad 2x + 1 = 0$
4. $\exists x \in \mathbb{N}/ \quad x \leq \pi$

5. $\cdots x \in \mathbb{R}: \quad x^2+2x+3=0$
6. $\cdots x \in \emptyset : 2=3$.

Exercise 7 1. *The function* $x \mapsto x^3 - x$ *is not positive for all* $x \in \mathbb{R}^+$, *only for* $x \geq 1$.
The negation : $\exists x = \frac{1}{2} \in \mathbb{R}^+ / \frac{1}{2^3} < \frac{1}{2}$.

2. $\forall x \in \mathbb{R}, \exists y = -x+1 \in \mathbb{R} : x+y > 0 \cdots (T)$.
3. $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x+y > 0 \cdots (F)$.
The negation : $\forall x \in \mathbb{R}, \exists y = -x-1 \in \mathbb{R} : x+y \leq 0$.
4. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y > 0 \cdots (F)$.
The negation : $\exists x = 1, \exists y = -2/ \quad x+y \leq 0 \cdots (T)$.
5. $\exists x = -1 \in \mathbb{R} : \forall y \in \mathbb{R}, \quad y^2 > x \cdots (T)$.
6. *(F). The negation :* $\exists x = 2 \in \mathbb{R} : \exists y = 3 \in \mathbb{R} : 2+3 \leq 2 \times 3$.
7. *(F).* $8 = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = 3+4 = 7$.
The negation : $\forall (x,y,z) \in \mathbb{R}^3, |x-y| \neq 3 \vee |y-z| \neq 4$ *or* $|x-z| \neq 8$.

Exercise 8 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a function. Give the negation of :*

1. $\exists M > 0 / \forall A > 0, \exists x \geq A / \quad f(x) \leq M$.
2. $\exists x \in \mathbb{R} / \quad , f(x) > 0 \wedge x > 0$.
3. $\exists \epsilon > 0 / \forall \eta > 0, \exists (x,y) \in I^2 / \quad (|x-y| \leq \eta \wedge |f(x)-f(y)| > \epsilon)$.

Exercise 9 1. f *is constant :* $\exists C \in \mathbb{R} / \quad \forall x \in \mathbb{R}, \quad f(x) = C$.

2. f *is not constant :*$\exists x \in \mathbb{R}, \exists y \in \mathbb{R} / \quad f(x) \neq f(y)$.
3. f *is increasing :* $\forall (x,y) \in \mathbb{R}^2 : (x \leq y \Rightarrow f(x) \leq f(y))$.
4. f *is bounded :* $\exists M \in \mathbb{R}, \exists m \in \mathbb{R} / \quad \forall x \in \mathbb{R}, \quad m \leq f(x) \leq M$.

Exercise 10 3. *Rewrite the contrapositive as : If* n *is odd, then* n^2 *is odd.*
Since n *is odd (hypothesis), we can let* $n = 2k+1$ *for some integer* k*, Then we are going to square it as the conclusion suggests, and show that it is odd.*

$$(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

$2k^2 + 2k$ *is also an integer. Then the square of an odd number is also an odd number.*

4. *Let us assume on the contrary that* $\sqrt{2}$ *is a rational number. Then, there exist two integers* p *and* q *such that*

$$\sqrt{2} = \frac{p}{q}, \quad q \neq 0; \tag{1}$$

where we may assume that p *and* q*, are co-prime i.e. have no common factors (if there are any common factors we cancel them in the numerator and denominator). Squaring (1)on both sides gives*

$$2 = \frac{p^2}{q^2} \quad i.e. \quad p^2 = 2q^2.$$

Thus p^2 *is even, then* p *is even (see 3.),* $p = 2k, k \in \mathbb{Z}$*, so* $2 = \frac{(2k)^2}{q^2}$, *i.e.* $2q^2 = 2 \times 2k^2$*, this means that* $q^2 = 2k^2$, q^2 *is even, then* q *has to be even, so* p *and* q *both even. We obtain that* 2 *is a common factor of* p *and* q*. But, this contradicts the fact that* p *and* q *have no common factor other than 1. This means that our supposition is wrong.*
Hence, $\sqrt{2}$ *is an irrational number.*

4. *Let us show by induction on the integer* $n \in \mathbb{N}^*$, *that the property* $P(n): \quad 2^{n-1} \leq n! \leq n^n$ *is true :*
Base case *: For* $n = 1$, *we have* $2^0 \leq 1! \leq 1$. *Thus,* $P(n)$ *is true for* $n = 1$.
Inductive Step *: Assume that* $P(n)$ *is true and show that* $P(n+1)$ *is true, which amounts to assuming that* $2^{n-1} \leq n! \leq n^n$ *and to show that* $2^n \leq (n+1)! \leq (n+1)^{n+1}$.
For $n \geq 1$, *we have* $n+1 \geq 2 \Rightarrow 2^n = 2 \times 2^{n-1} \leq (n+1)n! = (n+1)!$.
So $2^n \leq (n+1)! \cdots (i)$.
Also, we have $n \leq n+1$, *so* $n^n \leq (n+1)^n$,
and

$$(n+1)! = (n+1) \times n! \leq (n+1) \times n^n \leq (n+1) \times (n+1)^n = (n+1)^{n+1}.$$

i.e. $(n+1)! \leq (n+1)^{n+1} \cdots (ii)$
(i) *and* (ii) *shows that* $2^n \leq (n+1)! \leq (n+1)^{n+1}$ *and so* $P(n) \Rightarrow P(n+1)$.
In conclusion, by induction on the integer $n \in \mathbb{N}^*$, *we have* $2^{n-1} \leq n! \leq n^n$.