Algebra standard correction 2. First part.

Exercise 1

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

We prove that

1. $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$ *and*
2. $\overline{A} \cap \overline{B} \subset \overline{A \cup B}$

1. $\overline{A \cup B} \subset \overline{A} \cap \overline{B} \Leftrightarrow \forall x \in E, x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$.
   Let $x \in \overline{A \cup B} \Leftrightarrow x \in E$ *and* $x \notin A \cup B$, *then* $x \in E$ *and* $(x \notin A \wedge x \notin B)$ *which implies that* $(x \in E \wedge x \notin A)$ *and* $(x \in E \wedge x \notin B)$ *i.e.* $x \in \overline{A} \cap \overline{B}$.
2. $\overline{A} \cap \overline{B} \subset \overline{A \cup B} \Leftrightarrow \forall x \in E, x \in \overline{A} \cap \overline{B} \Rightarrow x \in \overline{A \cup B}$.
   Let $x \in \overline{A} \cap \overline{B} \Leftrightarrow (x \in E \wedge x \notin A)$ *and* $(x \in E \wedge x \notin B)$ *i.e* $x \in E \wedge (x \notin A \wedge x \notin B)$ *which implies that* $x \in E$ *and* $x \notin A \cup B$. *i.e.* $x \in \overline{A \cup B}$

Exercise 2 *Let E be a non-empty set. By contraposition we Prove that :*

1. $\forall A, B \in P(E) : A \neq B \Rightarrow A \cup B \neq A \cap B$.
   We have : $A \neq B \Leftrightarrow \exists x \in E / x \in A \wedge x \notin B$ *(or* $x \in B \wedge x \notin A$*).*
   We have $x \in A \subset A \cup B$ *so* $x \in A \cup B$ *and* $x \notin B \Rightarrow x \notin A \cap B$ *which implies that* $A \cup B \neq A \cap B$.
2. *By contraposition we Prove that :*
   $\forall A, B, C \in P(E) : B \neq C \Rightarrow (A \cap B \neq A \cap C \quad \text{and} \quad A \cup B \neq A \cup C)$.
   We have $B \neq C \Leftrightarrow \exists x \in E / \quad x \in B \wedge x \notin C$. *Two cases are then possible :*

a/ $x \in A$.
We have $x \in A \wedge x \in B \Rightarrow x \in A \cap B$ *and* $x \notin C \Rightarrow x \notin A \cap C$, *so we conclude that* $A \cap B \neq A \cap C$.

b/ $x \notin A$.
We have $x \notin A \wedge x \notin C \Rightarrow x \notin A \cup C$ *and* $x \in B \Rightarrow x \in A \cup B$, *so we conclude that* $A \cup B \neq A \cup C$.

Exercise 3 ***Symmetric difference***
Let A and B be two parts of a set E. We call the symmetric difference of A and B, and we denote $A \Delta B$, *the set defined by :*

$$A \Delta B = (A \cup B) \setminus (A \cap B).$$

1. *Show that* $A \Delta B = (A \setminus A \cap B) \cup (B \setminus A \cap B)$. *Let's proceed by double inclusion.*
   Let us show that $A \Delta B \subseteq (A \setminus A \cap B) \cup (B \setminus A \cap B)$. *Let* $x \in A \Delta B$. *By definition,* $x \in A \cup B$, *therefore* $x \in A$ *or* $x \in B$. *Suppose first that* $x \in A$, *the other case being symmetrical. By definition of the symmetric difference* $x \notin A \cap B$, *we therefore have* $x \in A \setminus A \cap B$. *By symmetry, if* $x \in B$, *we will have* $x \in B \setminus A \cap B$.
   Conclusion : *We have shown that for all* $x \in A \Delta B$, *we have* $x \in (A \setminus A \cap B)$ *or* $x \in (B \setminus A \cap B)$ *i.e* $A \Delta B \subseteq (A \setminus A \cap B) \cup (B \setminus A \cap B)$.
   Let us show that $(A \setminus A \cap B) \cup (B \setminus A \cap B \subseteq A \Delta B)$. *The proof is similar.*
2. $A \Delta E = E \setminus A = \overline{A}$.
   $A \Delta A = \emptyset$ *and* $A \Delta \emptyset = A$.
3. *Suppose* $A \Delta B = A \cap B$. *Prove by contradiction that :* $A = \emptyset, (B = \emptyset)$.
   Suppose $A \Delta B = A \cap B$. *To show that* $A = B = \emptyset$, *we just need to show that* $A = \emptyset$, *because A and B play symmetrical roles. Let us therefore show that* $A = \emptyset$.
   Suppose absurdly that there exists $a \in A$. *Two cases are then possible :*

a/ *1st case :* $a \in B$.
We have $a \in A \cap B = A \Delta B$. *However, by definition of the difference symmetric,* $a \notin A \cap B$, *a contradiction.*

a/ 2nd case : $a \notin B$.

We then have that $a \notin A \cap B$. *Since* $a \in A$, *we have that* $a \in A \cup B$, *and therefore* $a \in A\Delta B$. *Now,* $A\Delta B = A \cap B$, *therefore* $a \in A \cap B$, *therefore* $a \in B$, *a contradiction.*

Conclusion : *All cases lead to a contradiction, which means that there is no* $a \in A$, *and therefore* $A = \emptyset$.

4. *Let* $C \in P(E)$. *Show that* $A\Delta B = A\Delta C$ *if and only if* $B = C$.

If $B = C$, *then it is clear that* $A\Delta B = A\Delta C$.

Now suppose $A\Delta B = A\Delta C$, *and show that* $B = C$. *Again, we will proceed by double inclusion.*

Let us show that $B \subseteq C$. *Let* $b \in B$. *There are several possibilities :*

a/ If $b \in A$, *then it is in* $A \cap B$, *and therefore cannot be in* $A\Delta B$. *As* $A\Delta B = A\Delta C$ *by hypothesis,* $b \notin A\Delta C$. *Since* $b \in A$, *it must be in* $A \cap C$, *and* $b \in C$.

b/ If $b \notin A$, *then it is in* $A \cup B \setminus A \cap B = A\Delta B = A\Delta C$. *So* $b \in A \cup C$, *but* $b \notin A$, *therefore* $b \in C$.

In all cases, $b \in C$. *This being true for all* $b \in B$, *we indeed have* $B \subseteq C$.

Let us show that $C \subseteq B$. *The statement is symmetric in* B *and* C, *and* $B \subseteq C$.

Conclusion : $B = C$.