Chapter 1

Vector spaces

In this chapter $\mathbb{k}$ represents a field

1.1 Vector space

Definition 1.1. *A* ***vector space*** *over* $\mathbb{k}$ *is a non-empty set* E *endowed with two laws:*

- *an* ***internal composition law*** *called addition and denoted* " $+$ "

$$\begin{array}{rcl} + : E \times E & \longrightarrow & E \\ (x, y) & \longmapsto & x + y \end{array}$$

- *an* ***external composition law*** *called multiplication by a scalar and denoted by* " $\cdot$ "

$$\begin{array}{rcl} \cdot : \mathbb{k} \times E & \longrightarrow & E \\ (\lambda, x) & \longmapsto & \lambda \cdot x \end{array}$$

such that:

1. $(E, +)$ *is a* ***commutative group****, where the neutral element is denoted by* 0_E *and the symmetric of an element* x *of* E *will be denoted* $-x$;

2. *The external law must satisfy for all* $x \in E$ *and* $\alpha, \beta \in \mathbb{k}$:

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x;$$

3. for all $x, y \in E$ *and* $\alpha, \beta \in \Bbbk$:

$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x;$$

4. for all $x, y \in E$ *and* $\alpha, \beta \in \Bbbk$:

$$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y);$$

5. $1_{\Bbbk} \cdot x = x$.

Elementary property:

Let E be a $\Bbbk$ -vector space, then we have the following properties:

- $\forall x \in E, \ 0 \cdot x = 0_E$
- $\forall \alpha \in \Bbbk, \ \alpha \cdot 0 = 0_E$
- $\alpha \cdot x = 0_E \Leftrightarrow \alpha = 0_{\Bbbk}$ *or* $x = 0_E$;

Example 1.1. $(\mathbb{R}, +, .)$ *is a* $\mathbb{R}$*-vector space and* $(\mathbb{C}, +, .)$ *is a* $\mathbb{C}$*-vector space.*

Example 1.2. *We consider* $\Bbbk^n$ *the set of ordered sequences of* n *elements of* $\Bbbk$, *i.e.,* $(x_1, x_2, ..., x_n)$ *with* n *being a positive integer. Let* $x = (x_1, x_2, ..., x_n)$ *and* $x' = (x'_1, x'_2, ..., x'_n)$ *two elements of* $\Bbbk^n$ *and let* $\alpha \in \Bbbk$ *, we set:*
$x + x' = (x_1 + x'_1, x_2 + x'_2, ..., x_n + x'_n)$ *and* $\alpha.x = (\alpha.x_1, \alpha.x_2, ..., \alpha.x_n)$. *Equipped with these two laws, it is easy to verify that* $\Bbbk^n$ *is a* $\Bbbk$ *-vector space.*

Example 1.3. *The set* $V = F(\mathbb{R}, \mathbb{R})$ *of functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *equipped with the laws usual ways of adding functions, and multiplying a function by a scalar:* $(f + g)(x) = f(x) + g(x)$ *and* $(\alpha.f)(x) = \alpha.f(x)$, *is a* $\Bbbk$ *- vector space.*

1.1.1 Vector subspace

In this part, E will denote a $\mathbb{k}$-vector space.

Definition 1.2. *A subset F of E is called a vector subspace on $\mathbb{k}$ of E if*

(i) $\emptyset \neq F \subset E$,

(ii) F *is a* $\mathbb{k}$*-vector space.*

There is another technique to show that a subset F of E is vector subspace.

Theorem 1.1. *A subset F of E is called a* ***vector subspace*** *on $\mathbb{k}$ of E if the following condition hold :*

(i) $0_E \in F$;

(ii) $\forall x, y \in F,\ x + y \in F$;

(ii) $\forall \alpha \in \mathbb{k}, \forall x \in F,\ \alpha.x \in F.$

Theorem 1.2. *Let F be a nonempty subset of E, the following assertions are equivalence :*

- F *is a* ***vector subspace*** *over* $\mathbb{k}$,
- F *is* ***stable*** *for addition and for multiplication by a scalar .i.e*
  $\forall x,\ y \in F,\ x + y \in F;$ *and* $\forall \alpha \in \mathbb{k}, \forall x \in F, \alpha.x \in F.$
- $\forall x,\ y \in F,\ \forall \alpha,\ \beta \in \mathbb{k};\ \alpha.x + \beta.y \in F.$

Example 1.4. *(1). E and 0_E are vector sub-spaces of E .*

(2). $F = \{(x, y) \in \mathbb{R}^2 / x + y = 0\}$ *is a vector subspace of* $\mathbb{R}^2$ *over* $\mathbb{R}$ *because ,*

- $0_E = 0_{\mathbb{R}^2} = (0, 0) \in F \Rightarrow F \neq \emptyset$
- $\forall (x, y), (x', y') \in F, \forall \alpha, \beta \in \mathbb{R} : \alpha(x, y) + \beta(x', y') \in F$ *i.e* $(\alpha x + \beta x', \alpha y + \beta y') \in F$
  we have
  $(x, y) \in F \Rightarrow x + y = 0$ *and* $(x', y') \in F \Rightarrow x' + y' = 0$

$$\alpha x+\beta x'+\alpha y+\beta y'=\alpha(x+y)+\beta(x'+y')=\alpha(0)+\beta(0)=0$$

Then $\alpha(x,y)+\beta(x',y')\in F$*, so* F *is vector subspace of* E*.*

3. *The set* $F=\{(x,y)\in\mathbb{R}^2/x-y+1=0\}$ *is not a vector subspace of* $\mathbb{R}^2$ *because the zero vector* $0_{\mathbb{R}^2}$ *does not belong to* F*.*

1.1.2 Intersection and union of vector sub-spaces

Proposition 1.1. *The intersection of two vector sub-spaces is a vector subspace.*

Proof. Consider F_1 and F_2 two vector sub-spaces of E. First $0_E\in F_1$, because F_1 is a vector subspace of E. Similarly, $0_E\in F_2$. Thus, $0_E\in F_1\cap F_2$ and $F_1\cap F_2$ is therefore not empty. Given $x,y\in F_1\cap F_2$ and $\alpha,\beta\in\Bbbk$, we then have $\alpha x+\beta y\in F_1$ since F_1 is a vector subspace of E. Similarly, $\alpha x+\beta y\in F_2$. Thus, $\alpha x+\beta y\in F_1\cap F_2$. It follows that $F_1\cap F_2$ is a vector subspace of E. □

Lemma 1.1. *The intersection* $\cap_{i=1}^n F_i$ *of vector subspaces of a vector space* E *is a vector subspace of* E*.*

Remark 1.1. *In general, the union of two vector sub-spaces is not a vector subspace.*
Indeed, if we consider $E=\mathbb{R}^2$ *and the two vector sub-spaces*

$$D_1=\{(x,y)\in\mathbb{R}^2|y=0\} \text{ and } D_2=\{(x,y)\in\mathbb{R}^2|x=0\}.$$

Then $D_1\cup D_2$ *is not a vector subspace of* E*. For example,* $(\frac{1}{2},0)+(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$ *is the sum of an element of* D_1 *and an element of* D_2*, but is not in* $D_1\cup D_2$*.*

1.2 Generating families, Free families, Basis

- **Linear combination**

Definition 1.3. *For* $n\in\mathbb{N}^*$*, A linear combination of vectors* $u_1,u_2,...,u_n$ *of a* $\Bbbk$*-vector space* E*, is a vector which can be written* $V=\sum_{i=1}^n\lambda_i u_i$*. The elements* $\lambda_1,\lambda_2,...,\lambda_n\in\Bbbk$ *are called* ***coefficients*** *of the linear combination.*

Example 1.5. *In* $\mathbb{R}^3$*, the vector* $U = (3,3,1)$ *is a linear combination of vectors* $(1,1,0)$ *and* $(1,1,1)$ *because* $U = (3,3,1) = 2(1,1,0) + (1,1,1)$

Remark 1.2.
- *If* F *is a vector subspace of* E*, and* $u_1, u_2, ..., u_n \in F$*, then any linear combination* $\sum_{i=1}^{n} \lambda_i u_i$ *is in* F*.*
- *Let* $u_1, u_2, ..., u_n$*,* n *vectors of a* $\mathbb{k}$*-vector space* E*. One can always write* 0_E *as a linear combination of these vectors, because it suffices to take all zero coefficients of the linear combination.*
- *If* $n = 1$*, then* $V = \lambda_1 u_1$ *we say that* V *is* ***colinear*** *with* u_1*.*
- *In* $\mathbb{R}^2$*, the vector* $u = (2,1)$ *is not colinear with* $v = (1,1)$*.*

Notation

Given the vectors $u_1, u_2, ..., u_n$ of $\mathbb{k}$-vector space E, we denote $Vect(u_1, u_2, ..., u_n)$ or $\langle u_1, u_2, ..., u_n \rangle$the set of linear combination of $u_1, u_2, ..., u_n$. So we write :

$$\langle u_1, u_2, ..., u_n \rangle = Vect(u_1, u_2, ..., u_n) = \{u \in E \mid \exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{k}^n; u = \textstyle\sum_{i=1}^{n} \lambda_i u_i\}$$

- **Generating families**

We consider a nonempty family $A = (u_1, u_2, ..., u_n)$ of vectors of a $\mathbb{k}$ -vector space E with $n \in \mathbb{N}^*$.

Definition 1.4. *We say that* A ***generates*** E*, or that it is generator of* E *if and only if* $Vect(u_1, u_2, ..., u_n) = E$*. In other words, any vector of* E *is a linear combination of the elements of* A*.*

Example 1.6.
- $A = \{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$ ***generates*** $\mathbb{R}^3$*, because for all* $U = (x,y,z) \in \mathbb{R}^3$ *we have:* $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$
- *Let* $u_1 = (1,1,1), u_2 = (1,2,3)$ *two vectors of* $\mathbb{R}^3$

  We have:

$$(x,y,z) \in Vect(u_1,u_2) = \langle u_1, u_2 \rangle \Leftrightarrow (x,y,z) = \lambda_1(1,1,1) + \lambda_2(1,2,3) \Leftrightarrow x = \lambda_1 + \lambda_2,\ y = \lambda_1 + 2\lambda_2,\ z = \lambda_1 + 3\lambda_2$$

Then $\{u_1, u_2\}$ ***generates*** $\mathbb{R}^3$

- **Free families**

Definition 1.5. *We say that A is* ***free*** *if and only if the null vector* 0_E *is a linear combination of elements of A unique way. In other words:*

$$\forall \lambda_1, \lambda_2, ..., \lambda_n \in \Bbbk, \quad \sum_{i=1}^{n} \lambda_i u_i = 0_E \Rightarrow \lambda_1 = \lambda_2 = ...\lambda_i = 0_E.$$

Example 1.7. *The set* $A = \{u_1 = (1,0,1), u_2 = (0,2,2), u_3 = (3,7,1)\}$ *is free*

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, *we have*

$\sum_{i=1}^{n} \lambda_i u_i = \lambda_1(1,0,1) + \lambda_2 0,2,2) + \lambda_3(3,7,1) = 0_{\mathbb{R}^3} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

Remark 1.3. *We can use the following expressions:*

- *If A is free then we also say that the vectors* $(u_1, u_2, ..., u_n)$ *are linearly independent.*

- *If A is not free, we say that A is linked.*

- *A family of a single vector is* ***free*** *if and only if this vector is* ***non-zero.***

- **Basis**

Definition 1.6. *We say that A is a* ***basis*** *of a vector space E if it is* ***free*** *and* ***generating****. In other words, every vector of E is a linear combination of the elements of A in a unique way. So we have:*

$$\forall u \in E, \ \exists!(\lambda_1, \lambda_2, ..., \lambda_n) \in \Bbbk^n \quad u = \sum_{i=1}^{n} \lambda_i u_i$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ *are called the* ***coordinates*** *of the vector u in this basis A.*

1.3 Dimension of Vector spaces

finite type

Definition 1.7. *A vector space is said to be of* ***finite type*** *if it admits a finite generating family. In other words: if a vector space is generated by a finite family of vectors, it is said to be of* ***finite type*** *.*

Theorem 1.3. *In a finite dimensional vector space E, all **basis** have the same number of elements. This number denoted $dim(E)$ is called the **dimension** of E.*

Theorem 1.4. *Let A be a family of elements of E of finite dimension n. The following properties are equivalent:*

(i) A is a basis of E.

(ii) A is free and generates E.

(iii) A is free and $cardinal(A) = n$.

(v) A is the generator of E and $cardinal(A) = n$.

Example 1.8. *The set $A = \{u_1 = (1, 2), u_2 = (2, -1)\}$generates $\mathbb{R}^2$. What can we conclude? To show that A is a generating family, we look for two real λ_1, λ_2 such that: $for\ all u = (x, y) in \mathbb{R}^2$*
$U = \lambda_1 u_1 + \lambda_2 u_2$. After the calculation we will have $\lambda_1 = \frac{1}{5}(x + 2y), \lambda_2 = \frac{1}{5}(x - 2y)$ Which means that A generates $\mathbb{R}^2$. On the other hand, it is clear that A is free, of cardinal 2, so A is a basis of $\mathbb{R}^2$.

- We deduce that in a vector space E, any free family (or generator) whose number of elements is equal to the dimension of E is a basis.

Theorem 1.5. *Let F be a vector subspace of $E - \mathbb{k}$ vector space, we have*

- $dim(F) \leq dim(E)$
- $dim(F) = dim(E) \Leftrightarrow E = F$

Corollary 1.1. *(1)- Every vector space of finite type admits a finite basis, and all its bases have the same cardinality.*

In a vector space of dimension n, we have:

*(2)- Any **free** family has at most n elements.*

*(3)- Any **generating** family has at least n elements*

1.3.1 Rank of finite family of vectors

Definition 1.8. *Let E be a $\mathbb{k}$-vector space and $G = \{v_1, v_2, ..., v_m\}$ a family of m vectors of E. The* ***rank*** *of the family G noted $rank(G)$ is the dimension of the vector subspace $F = Vect(v_1, v_2, ..., v_m)$ generated by the vectors $v_1, v_2, ..., v_m$ i.e, $rank(G) = dim(F)$. or the largest number of linearly indepent vectors.*

Properties : Let E be a $\mathbb{k}$-vector space and $G = \{v_1, v_2, ..., v_m\}$ a family of vectors of E . So we have:

1 $0 \leq rank(G) \leq m$.

2 If $dim(E) = n$ (finite), then $rank(G) \leq n$.

3 $rank(G) = m$ if and only if G is free.

4 $rank(G) = 0$ if and only if all vectors of G are zero.

Example 1.9. *Let $G = \{v_1 = (2, 3), v_2 = (4, 2), v_3 = (-3, 4)\}$ be a family of the vector space $\mathbb{R}^2$. Determine the rank of G.*
It is clear that v_2 and v_3 are linearly independent. On the other hand, by solving the linear system $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, we get $2v_1 - v_2 - v_3 = 0$. The family G is therefore dependent. We deduce that $Vect(v_1, v_2, v_3) = Vect(v_1, v_2)$. So $rank(G) = 2$.

1.4 Complementary vector subspace

1.4.1 • Sum of two vector sub-spaces

Definition 1.9. *Let F_1 and F_2 be two vector sub-spaces of a $\mathbb{k}$-vector space E. We call* ***sum*** *of F_1 and F_2 the set noted $F_1 + F_2$, vectors which are the* ***sum*** *of a vector of F_1 and a vector of F_2:*

$$F_1 + F_2 = \{u : \ u = u_1 + u_2, \ u_1 \in F_1, \ u_2 \in F_2\}.$$

Remark 1.4. *We can characterize the vectors u of the sum $F_1 + F_2$, by:*

$$u \in F_1 + F_2 \;\Leftrightarrow\; \exists (u_1, u_2) \in F_1 \times F_2,\; u = u_1 + u_2$$

Proposition 1.2. *Let F_1 and F_2 be two vector sub-spaces of a $\Bbbk$-vector space E.*

(1)- $F_1 + F_2$ is a vector subspace of E.

(2)- $F_1 + F_2$ is the smallest vector subspace of E containing both F_1 and F_2.

Proof. (1) Consider F_1 and F_2 be two vector sub-spaces of E. First $0_E \in F_1$ because F_1 is a vector subspace of E. Similarly, $0_E \in F_2$ Thus, $0_E = 0_E + 0_E \in F_1 + F_2$ and $F_1 + F_2$ is therefore not empty. Let $x, y \in F_1 + F_2$ and $\alpha, \beta \in \Bbbk$. Since $x \in F_1 + F_2$, there are $x_1 \in F_1$ and $x_2 \in F_2$ such that : $x = x_1 + x_2 \; so \alpha x = \alpha(x_1 + x_2) = \alpha(x_1) + \alpha(x_2) \in F_1 + F_2$, because $\alpha(x_1) \in F_1$ and $\alpha(x_2) \in F_2$. Similarly for $y \in F_1 + F_2, \; weget \; \beta y = \beta(y_1 + y_2) = \beta(y_1) + \beta(y_2) \in F_1 + F_2$, because $\beta(y_1) \in F_1$ and $\beta(y_2) \in F_2$ with $y = y_1 + y_2$.
It follows that $\alpha x + \beta y = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) \in F_1 + F_2$.

(2) We first show that the set $F_1 + F_2$ contains both F_1 and F_2. Indeed, any element $u_1 \in F_1$ is written $u_1 = u_1 + 0_E$ with u_1 belonging to F_1 and 0_E belonging to F_2, because F_2 is a vector subspace of E. u_1 belongs to $F_1 + F_2$. The same for an element of F_2.
Now we show that if H is a vector subspace containing F_1 and F_2, then$F_1 + F_2 \subset H$. As $F_1 \subset H$. we therefore have, if $u_1 \in F_1$ then in particular $u_1 \in H$. Similarly, if $u_2 \in F_2$ then $u_2 \in H$.. Since H is a vector subspace, then $F_1 + F_2 \subset H$.

□

Example 1.10. *Determine $F + G$ where F and G be two vector sub-spaces of $\mathbb{R}^3$*
$F = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ and $G = \{(x, y, z) \in \mathbb{R}^3 : x = z = 0\}$
any element w of $F + G$ is written $w = u + v$ where u an element of F and v an element of G. For all $u \in F$ there exist $x \in \mathbb{R}$ such that $u = (x, 0, 0)$ and for all $v \in G$, there exist $y \in \mathbb{R}$ such that $v = (0, y, 0)$, so $w = u + v = (x, y, 0)$ is the sum of $(x, 0, 0)$ and $(0, y, 0)$.
Conversely, all element $w = (x, y, 0) = (x, 0, 0) + (0, y, 0)$ is a sum of an element of F and an element of G. Then $F + G = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$

Proposition 1.3. *4 (Grassmann formula). Let E be $\Bbbk-$ vector space of finite dimension , F_1 and F_2 be two vector sub-spaces of E, then :*

$$dimE = dimF_1 + dimF_2 - dim(F_1 \cap F_2)$$

For the existence of additional sub-spaces in finite dimension, the incomplete basis theorem says that in a finite dimensional vector space, any free family can be completed into a basis of the space. We immediately deduce the existence of supplementary ones.

1.4.2 • Direct sum of two vector sub-spaces

Proposition 1.4. *Let F_1 and F_2 be two vector sub-spaces of E. We say that the sum $F_1 + F_2$ is **direct** if any vector of $F_1 + F_2$ decomposes **uniquely** as the sum of an element of F_1 and an element of F_2*

$$E = F_1 \oplus F_2 \quad \text{then} \quad \forall w \in E, \quad \exists!\ u \in F_1 \quad \exists!\ \ v \in F_2 \quad \text{such that :} \quad w = u + v$$

Notation When F_1 and F_2 are in **direct sum,** we write $F_1 + F_2 = F_1 \oplus F_2$.

Definition 1.10. *Let F_1 and F_2 be two vector sub-spaces of E. We say that the sum $F_1 + F_2$ is **direct** $(E = F_1 \oplus F_2)$ if and only if*

- $F_1 \cap F_2 = 0_E$
- $E = F_1 + F_2$

Corollary 1.2. *Let E be $\Bbbk-$ vector space of finite dimension, then the following conditions are equivalent.*

(1) $E = F_1 \oplus F_2$

(2) $F_1 \cap F_2 = 0_E$ *and* $dimE = dimF_1 + dimF_2$

(3) $E = F_1 + F_2$ *and* $dimE = dimF_1 + dimF_2$

Remark 1.5.

(1) *If F and G are in direct sum, we say that F and G are supplementary sub-spaces in E.*

(2) *To say that an element can be uniquely expressed as the sum of an element in F and an element in G means that an element $w = u + v$ where $u \in F, v \in G$ and $w = u' + v'$ where $u' \in F, v' \in G$ then $u = u'$ and $v = v'$.*

(3) *In general, there is no uniqueness of the supplementary. In other words, for a vector subspace F_1 of a $\mathbb{k}$-vector space E, we can find many different supplementary F_2 such as $F_1 \oplus F_2 = E$. .*

Example 1.11. *(1)- Let $F = \{(x, 0) \in \mathbb{R}^2 / x \in \mathbb{R}\}$ and $G = \{(0, y) \in \mathbb{R}^2 / y \in \mathbb{R}\}$ two sub-spaces of $\mathbb{R}^2$.*

$\mathbb{R}^2 = F \oplus G$ because $F \cap G = \{0_{\mathbb{R}^2}\}$ and if any vector of $\mathbb{R}^2$ decompose uniquely as $(x, y) = (x, 0) + (0, y)$, then $\mathbb{R}^2 = F + G$,

(2)- We show that there is no uniqueness of the supplementary of a sub-space.

Let's keep $F = \{(x, 0) \in \mathbb{R}^2 / x \in \mathbb{R}\}$ and $G' = \{(x, x) \in \mathbb{R}^2 / x \in \mathbb{R}\}$

we have $\mathbb{R}^2 = F \oplus G'$

show that $F \cap G = \{0_{\mathbb{R}^2}\} = (0, 0)$.

If $(x, y) \in F \cap G'$, then $(x, y) \in F$ so $y = 0$ and $(x, y) \in G$ so $x = y$ then $(x, y) = (0, 0)$.

Show that $\mathbb{R}^2 = F + G'$

Let $u = (x, y) \in \mathbb{R}^2$. Find $v \in F$ and $w \in G'$ such that $u = v + w$

$(x_1, y_1) \in F$ so $y_1 = 0$ and $(x_2, y_2) \in G'$ then $x_2 = y_2$. It's about finding x_1 and x_2 such that $(x, y) = (x_1, 0) + (x_2, x_2)$ then $(x, y) = (x_1 + x_2, x_2)/\ x = x_1 + x_2, y = x_2$

Finally $(x, y) = (x - y, 0) + (y, y)$

Exercise 1. *Let $\mathbb{R}^3$ be the vector space on the field $\mathbb{R}$, $G = [(1, 1, 0), (0, 0, 1), (1, 1, 1)]$, be a vector subspace of $\mathbb{R}^3$ and let the set F be defined as : $F = \{(x, y, z) \in \mathbb{R}^3 / 2x + y - z = 0\}$*

1. *Show that F is a vector subspace of $\mathbb{R}^3$.*

2. *Find a basis for each of :* $F,\ G,\ F \cap G,\ F + G$, *and give their dimensions.*

3. *Is* $\mathbb{R}^3 = F \oplus G$?